Oct. 16/17

LINEAR PROG

Chapter 6 - Sensitivity Analysis and Duality

$$\text{Max } z = c_1x_1 + c_2x_2$$
$$\text{s.t. } a_{11}x_1 + a_{12}x_2 \le b_1$$
$$a_{21}x_1 + a_{22}x_2 \le b_2$$
$$x_1, x_2 \ge 0$$

Ex. Giapetto Problem:

$$\text{Max } z = 3x_1 + 2x_2$$
$$\text{s.t. } 2x_1 + x_2 \le 100$$
$$x_1 + x_2 \le 80$$
$$x_1 \le 40$$
$$x_1, x_2 \ge 0$$

$X^* = \begin{bmatrix} 20 \\ 60 \end{bmatrix}$

$z^* = 180$

(x) (y)

$3x_1 + 2x_2 = \text{Constant}$

$x_2 = -3/2\, x_1 + \frac{\text{Constant}}{2}$ $\longrightarrow$

$(y = mx + b)$

1

Slope : $-\frac{c_1}{2} > -1$

$-c_1 > -2$

$c_1 < 2$

$X^* = \begin{bmatrix} 0 \\ 80 \end{bmatrix}$

2

slope $= -\frac{c_1}{2} < -2$

$-c_1 < -4$

$c_1 > 4$ $\rightarrow$ $X^* = \begin{bmatrix} 40 \\ 20 \end{bmatrix}$

$2 \le c_1 \le 4 \rightarrow X^* \begin{bmatrix} 20 \\ 60 \end{bmatrix}$

$c_1 = 2$; $z^* = 2(20) + 2(60) = 160$

$c_1 = 4$ $z^* = 4(20) + 2(60) = 200$

$$\begin{cases} 2x_1 + x_2 = 100 + \Delta \\ x_1 + x_2 = 80 \end{cases} \rightarrow \quad x_1 + x_1 + x_2 = 100 + \Delta$$

$x_1 = 100 - 80 + \Delta = 20 + \Delta$

$x_2 = 80 - (20 + \Delta) = 60 - \Delta$

$x_1 + x_1 + x_2 = x_1 + 80 > 120$

① $b_1 > 120$ $\quad x_1 > 40$, $\bar{x}$ is not optimal

② $b_1 < 80$ $\quad x_1 < 0$, $\bar{x}$ is not optimal

③ $80 \le b_1 \le 120$ $\quad \bar{x}$ is still optimal

$b_1 \longrightarrow 100 + \Delta$, $\quad 80 \le 100 + \Delta \le 120$

$-20 < \Delta \le 20$

$Z^* = 3(20 + \Delta)$

$= 2(60 - \Delta)$

$= 60 + 3\Delta + 120 - 2\Delta$

$= 180 + \Delta$

$\text{Max } Z = \mathbb{C}\mathbb{X}$

$\text{s.t. } A_{m \times n}\mathbb{X} = \mathbb{b}$

$\mathbb{X} \ge 0$

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix}$

$\mathbb{C} = [C_1, C_2, \ldots C_n]$

$\mathbb{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbb{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$BV = \{x_{BV_1}, x_{BV_2}, \ldots x_{BV_m}\}$

$NBV = \{x_{NBV_1}, x_{NBV_2}, \ldots x_{NBV(n-m)}\}$

$\mathbb{C}_{BV} = \{C_{xBV_1}, C_{xBV_2}, \ldots, C_{xBV_m}\}$

$\mathbb{C}_{NBV} = \{C_{xNBV_1}, C_{xNBV_2}, \ldots, C_{xNBV(n-m)}\}$

$B = [a_{xBV_1}, a_{xBV_2}, \ldots a_{xBV_m}]$

$N = [a_{xNBV_1}, a_{xNBV_2}, a_{xNBV(n-m)}]$

$$
\begin{array}{lllllll}
Z & + 5x_2 & & + 10 s_2 & + 10 s_3 & = 280 \\
 & - 2x_2 & + s_1 & + 2s_2 & - 8 s_3 & = 24 \\
 & - 2x_2 + x_3 & & + 2s_2 & - 4 s_3 & = 8 \\
x_1 & + 1.25x_2 & & - 0.5 s_2 & + 1.5 s_3 & = 2
\end{array}
$$

$BV = \{s_1, x_3, x_1\}$

$NBV = \{x_2, s_2, s_3\}$

$C_{BV} = \{0, 20, 60\}$

$C_{NBV} = \{30, 0, 0\}$

$$
B = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix}, \quad \begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} = X_{BV}, \quad b = \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix}
$$

$$
N = \begin{array}{c} \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ C_{x_2} & C_{s_2} & C_{s_3} \end{array} \end{array}, \quad \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix} = X_{NBV}
$$

$$
\begin{aligned}
Z &= C_{BV} X_{BV} + C_{NBV} X_{NBV} \\
& B X_{BV} + N X_{NBV} = b \\
& X_{BV} \geq 0, \quad X_{NBV} \geq 0
\end{aligned}
$$

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$Z = \mathbb{C}\mathbb{X}$

s.t. $A\mathbb{X} = \mathbb{b}$

$\mathbb{X} \geq 0$

BV - NBV

$Z = \mathbb{C}_B \mathbb{X}_B + \mathbb{C}_N \mathbb{X}_N$

$B\mathbb{X}_B + N\mathbb{X}_N = \mathbb{b}$

$\mathbb{X}_B \geq 0, \quad \mathbb{X}_N \geq 0$

$B^{-1}(B\mathbb{X}_B + N\mathbb{X}_N) = B^{-1}\mathbb{b}$

$\mathbb{X}_B + B^{-1}N\mathbb{X}_N = B^{-1}\mathbb{b}$

$Z = \mathbb{C}_B(B^{-1}\mathbb{b} - B^{-1}N\mathbb{X}_N) + \mathbb{C}_N\mathbb{X}_N$

$= \mathbb{C}_B B^{-1}\mathbb{b} + (C_N - C_B B^{-1} N)\mathbb{X}_N$

$Z^* = \mathbb{C}_B B^{-1}\mathbb{b}$

* Next midterm to have question similar to "Multiplying the constraints by B^{-1} yields" slide.

Ex. $Z + (\mathbb{C}_B B^{-1} N - \mathbb{C}_N)\mathbb{X}_N = \mathbb{C}_B B^{-1}\mathbb{b}$

$x_B + B^{-1}N\mathbb{X}_N = B^{-1}\mathbb{b}$

$BV = \{s_1, x_3, x_1\}$

$NBV = \{x_2, s_2, s_3\}$

$a_{x_2} = \begin{bmatrix} 6 \\ 2 \\ 1.5 \end{bmatrix} \qquad \mathbb{C}_B = [0, 20, 60]$

$B = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix}$

$B^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}$

Z	x_1	x_2	x_3	s_1	s_2	s_3	
1	0	5	0	0	10	10	280
	0	-2	0	1	2	-8	24
	0	-2	1	0	2	-4	8
	1	1.25	0	0	-0.5	1.5	2