Recap :

- Fourier Transform : $f: \mathbb{R} \to \mathbb{R}$

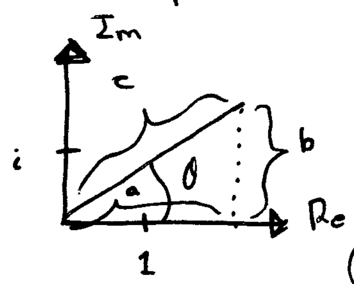
  $F[f](\omega) = \int_{\mathbb{R}} f(x)\, e^{-i\omega x}\, dx$

  instead of $[0, \infty)$ → complex exponential

  1) Linear : $\mathcal{F}[f + cg] = \mathcal{F}[f] + c\,\mathcal{F}[g]$
  2) Invertible : $\mathcal{F}^{-1}[\mathcal{F}[f]] = f$

     if $F[f] = \mathcal{F}[g] \to f = g$
  3) NOT multiplicative : $\mathcal{F}(fg) \neq \mathcal{F}[f] \cdot \mathcal{F}[g]$

- Complex numbers $\mathbb{C}$



$z = a_i + b_i$

$(i = -1)$

$z = a + bi = re^{i\theta}$

$r = \sqrt{a^2+b^2}$

$\theta = \arctan(b/a)$

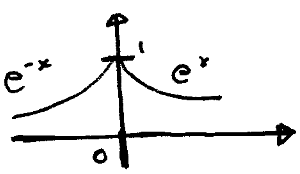
$a = r\cos\theta$

$b = r\sin\theta$

$e^{i\theta} = a\cos\theta + i\sin\theta$

Today: Examples of $\mathcal{F}$

Ex: Find $\mathcal{F}[f]$, $f(x) = e^{-|x|}$



$\mathcal{F}[f](\omega) = \int_{\mathbb{R}} f(x) e^{-i\omega x}\, dx$

$= \int_{-\infty}^{0} e^{x} e^{i\omega x}\, dx + \int_{0}^{\infty} e^{-x} e^{-i\omega x}\, dx$

$= \int_{-\infty}^{0} e^{(1-i\omega)x}\, dx + \int_{0}^{\infty} e^{-(1+i\omega)x}\, dx$

$$= \frac{e^{(1-i\omega)x}}{1-i\omega}\Bigg|_{-\infty}^{0} + \frac{e^{-(1+i\omega)x}}{-(1+i\omega)}\Bigg|_{0}^{+\infty}$$

$$= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{2}{1+\omega^2}$$

$(1-i\omega)(1+i\omega) = 1 + i\omega - i\omega - i^2\omega^2 = 1 + \omega^2$

<u>Ex</u>: $\lim_{x\to\pm\infty} f(x) = 0$ Find $\mathcal{F}[f']$ in terms of $\mathcal{F}[f]$

$$\mathcal{F}[f'](\omega) = \int_{\mathbb{R}} \underbrace{f'(x)}_{dv}\, \underbrace{e^{-i\omega x}}_{u}\, dx$$

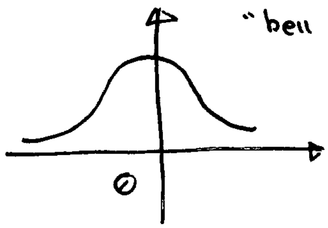
$$= f(x)e^{-i\omega x}\Big|_{-\infty}^{+\infty} + i\omega \underbrace{\int_{\mathbb{R}} f(x)\, e^{-i\omega x}\, dx}_{\mathcal{F}[f](\omega)}$$

$\rightarrow$ $\underline{\mathcal{F}[f'](\omega) = i\omega\, \mathcal{F}[f](\omega)}$ $\rightarrow$ important !!

and... $\rightarrow$ $\mathcal{F}[f''](\omega) = i\omega\mathcal{F}[f'](\omega) = (i\omega)^2\mathcal{F}(f)(\omega)$

thus, $\rightarrow$ $\mathcal{F}[f^{(n)}](\omega) = (i\omega)^n\mathcal{F}[f](\omega)$

<u>Ex</u>: Find $\mathcal{F}[g]$, $g(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-x^2}{2\pi^2}}$



"bell distribution" $\sigma > 0$ (given parameter)

$g(x)$ = Gaussian distribution

mean = 0

std. dev σ

If we did

$$\mathcal{F}[g](\omega) = \int_{\mathbb{R}} g(x)e^{-i\omega x}\, dx$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} g(x)\, e^{-i\omega x}\, dx$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} \underbrace{e^{(-x^2/2\sigma^2)} e^{-i\omega x}}_{\text{NOT integrable}}\, dx$$

NOTE: $g'(x) = -\left(\frac{x}{\sigma^2}\right) \cdot \underbrace{\left(\frac{1}{\sigma\sqrt{2\pi}}\right) e^{-x^2/2\sigma^2}}_{g(x)} = \dfrac{-x}{\sigma^2}\, g(x)$

Take $\mathcal{F}$ both sides:

$$\mathcal{F}[g'](\omega) = i\omega\, \mathcal{F}[g](\omega)$$
$$\mathcal{F}\left[\frac{-x}{\sigma^2} g(x)\right](\omega) = \int_{\mathbb{R}} \frac{-x}{\sigma^2}\, g(x)\, e^{-i\omega x}$$
$$= \frac{1}{i\sigma^2}\, \frac{dG(\omega)}{d\omega}$$

$G = \mathcal{F}[g]$

$\rightarrow$ $i\omega G(\omega) = \dfrac{1}{i\sigma^2}\, G'(\omega)$

$\rightarrow$ $-\omega\sigma^2 = \dfrac{G'(\omega)}{G(\omega)} = \dfrac{d}{d\omega} \ln|G(\omega)|$

$\rightarrow$ $\therefore\ G(\omega) = e^{-\frac{\omega^2\sigma^2}{2}} \cdot a$ a = <u>const.</u>

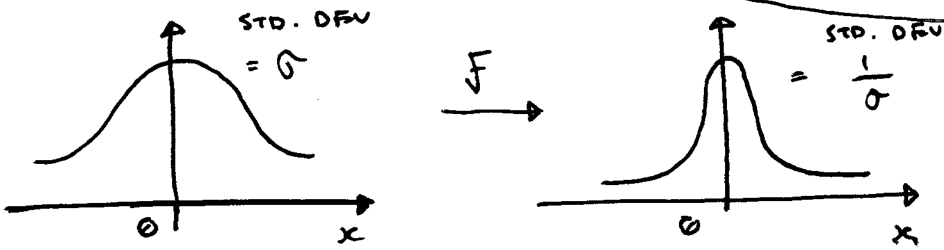
$\rightarrow \quad G(\omega) = c e^{\frac{-\omega^2 \sigma^2}{2}}$ $\qquad C = \pm a$ const.

$\rightarrow \quad G(0) = \mathcal{F}[g](0) = \int_{\mathbb{R}} g(x) e^{-i \cdot 0 \cdot x} dx = 1$

$\hookrightarrow \quad G(\omega) = \mathcal{F}[g](\omega) = \underbrace{e^{\frac{-\omega^2 \sigma^2}{2}}}$

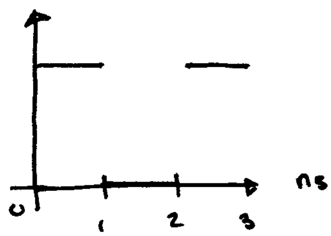
Gaussian mean $= 0$

std. dev. $\frac{1}{\sigma}$



EX Each signal has transmission time 1ns.

Your signal is "101". Find its $\mathcal{F}$.



$$f(t) = \begin{cases} 1 & \text{if } t \in (0,1) \cup (2,3) \\ 0 & \text{if not} \end{cases}$$

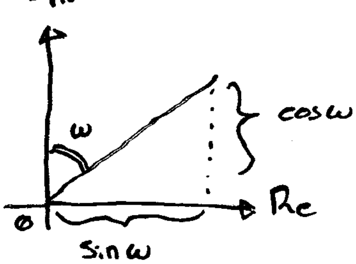
We need to find $\mathcal{F}[f]$

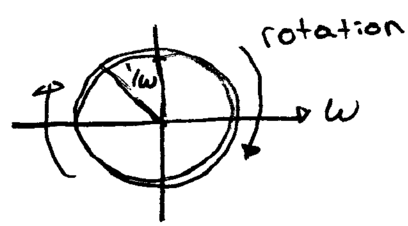
$$\begin{aligned} F[f](\omega) &= \int_{\mathbb{R}} f(t) e^{-i\omega t} dt \\ &= \int_0^1 e^{-i\omega t} dt + \int_2^3 e^{-i\omega t} dt \\ &= \frac{e^{-i\omega t}}{-i\omega}\Big|_0^1 + \frac{e^{-i\omega t}}{-i\omega}\Big|_2^3 \\ &= \frac{1 - e^{-i\omega}}{i\omega} + \frac{e^{-2i\omega} - e^{-3i\omega}}{i\omega} \end{aligned}$$

We plot $\frac{e^{-i\omega}}{i\omega} = \frac{-i}{\omega} \underbrace{(\cos\omega - i\sin\omega)}_{= e^{-i\omega}}$

$\frac{i^2}{i} = \frac{-1}{i} \longrightarrow i = \frac{-1}{i}$ $\qquad = \frac{1}{\omega}(-\sin\omega - i\cos\omega)$

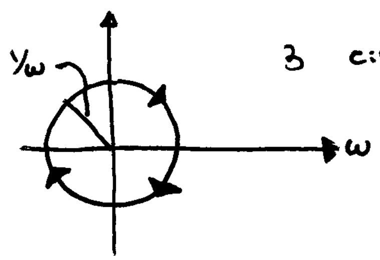
$\frac{1}{i} = -1$ $\qquad = \frac{-1}{\omega}(\sin\omega + i\cos\omega)$





period $= \frac{2\pi}{\omega}$

ω = frequency



3 circular waves : Freq. ω, 2ω, 3ω

Now we take $\mathcal{F}^{-1}$:

$$\mathcal{F}^{-1}\left[\frac{1-e^{i\omega}+e^{-2i\omega}-e^{-3i\omega}}{i\omega}\right](t)$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1-e^{-i\omega}+e^{-2i\omega}-e^{-3i\omega}}{i\omega} e^{i\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{\mathbb{R}} \frac{e^{i\omega t}}{i\omega}\, d\omega - \int_{\mathbb{R}} \frac{e^{-i\omega}e^{i\omega t}}{i\omega}\, d\omega + \int_{\mathbb{R}} \frac{e^{-2i\omega}e^{i\omega t}}{i\omega}\, d\omega \cdots\right.$$

$$\cdots - \int_{\mathbb{R}} \frac{e^{-3i\omega}e^{i\omega t}}{i\omega}\, d\omega$$

<u>Integrate</u> : $\int_{\mathbb{R}} \frac{e^{i\omega t}}{i\omega}\, d\omega$

$\sim$ like to integrate $\int_{\mathbb{R}} x^{-1} e^{cx}\, dx$

Integrate $\int_{\mathbb{R}} \frac{(1-e^{i\omega})}{i\omega} e^{i\omega t}\, dt$

$$= \mathcal{F}^{-1}\left[\frac{1}{i\omega}\cdot 1\right] \qquad \begin{cases} 1 & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

$$= \mathcal{F}\left[\int_0^t \frac{\delta_0(t)}{i\omega}\right] \qquad = \text{antiderivative of } \delta_0(t)$$

$$\mathcal{F}[f'](\omega) = i\omega\,\mathcal{F}[f](\omega)$$

$$\mathcal{F}[\int f](\omega) = \frac{1}{i\omega}\mathcal{F}[f](\omega) \qquad = \text{antiderivative of}$$

$$(\delta_0(t) - \delta_1(t) + \delta_2(t) - \delta_3(t))$$

$$\int_0^T \delta_0(t) - \delta_1(t) = \begin{cases} 1 & \text{between } (0,1) \\ 0 & \text{elsewhere} \end{cases}$$

$$\int_0^T \delta_2(t) - \delta_3(t) = \begin{cases} 1 & \text{between } (2,3) \\ 0 & \text{elsewhere} \end{cases}$$

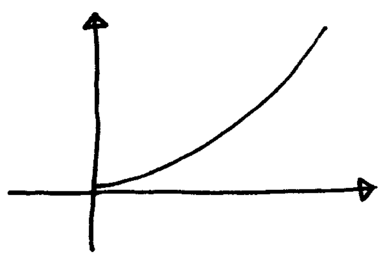
MAR. 5/19

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega}\, dt$$

$\hookrightarrow f$ $\qquad \omega = 2\pi f = \frac{2\pi}{T}$

$$f_1(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0 \end{cases} \qquad f_2(t) = \begin{cases} 0 & t < 0 \\ e^{t} & t \geq 0 \end{cases}$$

$$\begin{aligned}
\mathcal{F}(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{0} 0\, dt + \int_{0}^{\infty} e^{-t} e^{-j\omega t}\, dt \\
&= \int_{0}^{\infty} e^{-t(1+j\omega)}\, dt \\
&= \left. \frac{e^{-t(1+j\omega)}}{-(1+j\omega)} \right]_0^{\infty} \\
&= \frac{0 - 1}{-(1+j\omega)} = \frac{1}{1+j\omega}
\end{aligned}$$



$0 : t < 0$

$e^{t} : t \geq 0$

no $\mathcal{F}(\omega)$

$$\underbrace{f(t) = \delta(t)}$$

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, f(t) = f(t_0)$$

$$\int_{-\infty}^{\infty} \underbrace{\delta(t-4)}_{t_0 = 4} \underbrace{e^{-t}}_{f(4) = e^{-t}}\, dt$$

$$\int_{-\infty}^{\infty} e^{(t-1)} \cos\left(\pi/2\,(t-5)\right) \delta(t-3)\, dt$$

$$\int_{-\infty}^{\infty} f(t) \qquad\qquad \delta(t - t_0)$$

$$f(t_0)$$

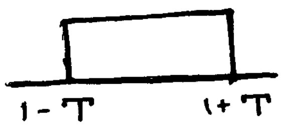
$$= e^{(3-1)} \cos\left(\pi/2\,(3-5)\right)$$

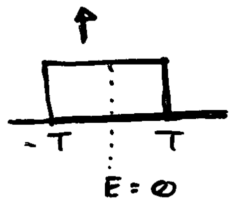
$$= e^{2} \cos\left(\tfrac{\pi}{2}(-2)\right) \rightsquigarrow -e^{2}$$

rectangular

$-T \le t \le T$

$$f(t) = \begin{cases} 1 & 1-T \le t \le 1+T \\ 0 & \text{else} \end{cases}$$





← Shifted rectangular pulse

$$F(\omega) = \int_{1-T}^{1+T} (1)\, e^{-j\omega t}\, dt$$

$$= \frac{-1 \cdot e^{-j\omega t}}{j\omega} \Big|_{1-T}^{1+T}$$

$$= \frac{-1}{j\omega}\left(e^{-j\omega(1+T)} - e^{-j\omega(1-T)} \right)$$

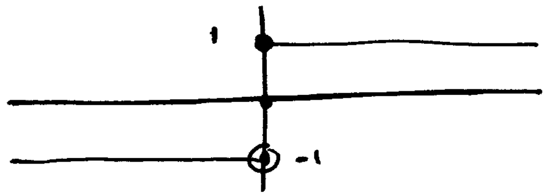
$$= \frac{-e^{j\omega}}{j\omega}\left(e^{-j\omega t} - e^{j\omega t} \right)$$

$$= \frac{2 \times e^{j\omega}}{\omega}\left(\frac{e^{j\omega T} - e^{-j\omega T}}{2j} \right)$$

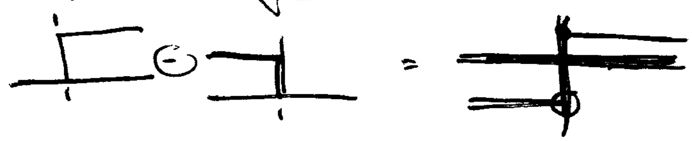
$$= \frac{2e^{-j\omega}}{\omega} \sin(\omega T)$$

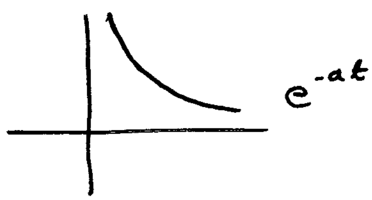
$$\frac{2 \sin(\omega t)}{\omega}$$

$$x(t) = sgn(t) = \begin{cases} -1 & t < 0 \\ 0 & t = 0 \\ 1 & t > 0 \end{cases}$$



$$sgn(t) = \underbrace{u(t)} - \underbrace{u(-t)}$$





$$= \lim_{a \to 0} e^{-at} u(t)$$

$$u(t) = \lim_{a \to 0} e^{at} u(t)$$

$$F \left(sgn(t) = \lim_{a \to 0} \left[e^{-at} u(t) - e^{at} u(t) \right] \right)$$

$$f_1(t) = e^{-t} u(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & \geq 0 \end{cases}$$

$$F(sgn(t)) = \lim_{a \to 0} \left[\frac{1}{(a+jw)} - \frac{1}{(a-jw)} \right]$$

$$\lim_{a \to 0} \left[\frac{(a-jw) - (a+jw)}{(a+jw)(a-jw)} \right]$$

$$(A+B)(A-B) = A^2 - B^2$$

$$\lim_{a \to 0} \left[\frac{a - jw - a - jw}{a^2 - (jw)^2} \right]$$

$$\lim_{a \to 0} \left[\frac{-2jw}{a^2 + w^2} \right] \longrightarrow \left[\frac{-2jw}{w^2} \right] = -2jw$$

$$= \frac{2}{jw}$$

$$\boxed{\mathcal{F}(sgn(t)) \longrightarrow \; = \frac{2}{jw}}$$

Recap:

- examples of $\mathcal{F}$

1) Find $\mathcal{F}[f]$, $f(x) = e^{-|x|}$

$$\mathcal{F}[f](\omega) = \int_{\mathbb{R}} f(x) e^{-i\omega x}\, dx \qquad \left(\int_{\mathbb{R}} e^{-|x|} \sin \omega x = 0\right)$$
$$= \int_{\mathbb{R}} e^{-|x|}(\cos \omega x + \cancel{i \sin \omega x})\, dx$$
$$= 2\int_0^{+\infty} e^{-x} \cos \omega x\, dx = \frac{2}{1+\omega^2}$$

2) $$\mathcal{F}[f']\omega = \int_{\mathbb{R}} f'(x) e^{-i\omega x}\, dx$$
$$= \underbrace{\cancel{f(x) e^{-i\omega x}\Big|_{-\infty}^{+\infty}}}_{=0} + i\omega \underbrace{\int_{\mathbb{R}} f(x) e^{-i\omega x}\, dx}_{\mathcal{F}[f](\omega)}$$
$$= i\omega \mathcal{F}[f](\omega)$$

more in general: $\mathcal{F}[f^{(n)}](\omega) = (i\omega)^n \mathcal{F}[f](\omega)$

3) Signal "101" $\left(f(t) = \begin{cases} 1 & \text{on } (0,1) \cup (3,4) \\ 0 & \text{elsewhere} \end{cases} \right)$

Today: Solve PDE's with $\mathcal{F}$

1) Take Fourier ($\mathcal{F}$)
2) Do computations
3) Take $\mathcal{F}^{-1}$

Ex Solve $u_t = k u_{xx}$ $(k > 0)$ subject to

$u(x,0) = f(x)$ $\quad \lim_{x \to \pm\infty} u(x,t) = 0$

$\underline{x \in \mathbb{R}}$ & $t \in [0, +\infty]$

1) Take $\mathcal{F}$:

$$\mathcal{F}[u_t](\omega, t) = \int_{\mathbb{R}} u_t(x,t) e^{-i\omega x} dx$$
$$= \frac{\partial}{\partial t} \int_{\mathbb{R}} u(x,t) e^{-i\omega x}\, dx = \hat{u}_t(\omega, t) \quad \& \quad \hat{u} = \mathcal{F}[u]$$
$$\mathcal{F}[u_{xx}](\omega, t) = (i\omega)^2 \mathcal{F}[u](\omega, t) = -\omega^2 \hat{u}(\omega, t)$$
$$\rightarrow \hat{u}_t(\omega, t) = k \cdot -\omega^2 \hat{u}(\omega, t) \qquad \underline{\text{ODE in } \hat{u}}$$

2) Solution of ODE: $\hat{u}(\omega, t) = \underbrace{A(\omega)}_{\text{A may depend on } \omega} e^{-k\omega^2 t}$

$\rightarrow \hat{u}(\omega,0) = \int_{\mathbb{R}} \underbrace{u(x,0)}_{= f(x)} e^{-i\omega x}\,dx$

$= \hat{f}(\omega)$

$(\hat{f} = \mathcal{F}[f])$

$\rightarrow \hat{u}(\omega,t) = \hat{f}(\omega)e^{-k\omega^2 t}$

$\hat{u}(\omega,0) = A(\omega)\cancel{e^{-k\omega^2(0)}}_{=1}$

$= \hat{f}(\omega)$

3) Take $\mathcal{F}^{-1}$

Rule : $\underbrace{\mathcal{F}[f*g]}_{\text{convolution}} = \mathcal{F}[f]\,\mathcal{F}[g]$

$f*g(x) = \int_{\mathbb{R}} f(z)g(x-z)\,dz = \int_{\mathbb{R}} f(x-z)g(z)\,dz$

$\hat{u}(\omega,t) = \hat{f}(\omega)e^{-k\omega^2 t}$

$= \hat{f}(\omega)\,\mathcal{F}[\underbrace{\mathcal{F}^{-1}[e^{-k\omega^2 t}]}_{\text{to find}}]$

$\mathcal{F}^{-1}[g](x) = \frac{1}{2\pi}\int_{\mathbb{R}} g(\omega)e^{i\omega x}\,d\omega$

$\mathcal{F}^{-1}[e^{-k\omega^2 t}] = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-k\omega^2 t}e^{i\omega x}\,d\omega$

$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-(k\omega^2 t - i\omega x)}\,d\omega$

$k\omega^2 t - i\omega x$

$= (\omega\sqrt{kt})^2 \rightarrow = 2\cdot\omega\sqrt{kt}\,\frac{ix}{2\sqrt{kt}} + \left(\frac{ix}{2\sqrt{kt}}\right)^2 - \left(\frac{ix}{2\sqrt{kt}}\right)^2$

$= \left(\omega\sqrt{kt} - \frac{ix}{2\sqrt{kt}}\right)^2 - \left(\frac{ix}{2\sqrt{kt}}\right)^2$

$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\underbrace{\left(\omega\sqrt{kt} - \frac{ix}{2\sqrt{kt}}\right)^2}_{=y} + \underbrace{\left(\frac{ix}{2\sqrt{kt}}\right)^2}_{= -\frac{x^2}{4\sqrt{kt}}}}\,d\omega$

$y = \omega\sqrt{kt} - \frac{ix}{2\sqrt{kt}}$

$dy = \sqrt{kt}\,d\omega \rightarrow d\omega = dy/\sqrt{kt}$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-y^2} e^{-x^2/4\kappa t}\,\frac{dy}{\sqrt{\kappa t}}$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{4\kappa t}} \cdot \left(\frac{1}{\sqrt{\kappa t}}\right) \underbrace{\int_{\mathbb{R}} e^{-y^2}\,dy}_{= 2\int_0^{+\infty} e^{-y^2}\,dy \,=\, \sqrt{\pi}}$$

$$= \left(\frac{1}{2\sqrt{\pi\kappa t}}\right) e^{-\frac{x^2}{4\kappa t}}$$

$$\underbrace{G(x,t)}_{\text{"heat kernel"}} = \mathcal{F}^{-1}\left[e^{-\kappa\omega^2 t}\right] = \frac{e^{-x^2/4\kappa t}}{2\sqrt{\pi\kappa t}}$$

→ $\hat{u}(\omega,t) = \hat{f}(\omega)\,\mathcal{F}[G](\omega,t)$

→ Solution :

$$u(x,t) = (f * (x)(x,t))$$

$$= \int_{\mathbb{R}} f(x-z)\,\frac{e^{-x^2/4\kappa t}}{2\sqrt{\pi\kappa t}}\,dz$$

$$\left.\begin{aligned} u_t &= \kappa u_{xx} \\ u(x,0) &= f(x) \end{aligned}\right\} \text{ Solution } \quad u = f * G$$

Ex Solve $u_{tt} = c^2 u_{xx}$ subject to $u(x,0) = f(x)$

$\lim_{x\to\pm\infty} u(x,t) = 0$ $\qquad u_t(x,0) = g(x)$

1) Take $\mathcal{F}$:

$\mathcal{F}[u_{tt}] = \partial^2/\partial t^2\,\hat{u}$

$\left(= \int_{\mathbb{R}} \partial^2/\partial t^2\, u(x,t)\,e^{-i\omega x}\,dx = \frac{\partial^2}{\partial t^2}\int_{\mathbb{R}} u(x,t)\,e^{-i\omega x}\,dx\right)$

$\mathcal{F}[u_{xx}] = -\omega^2\hat{u}$ $\qquad \underbrace{\hat{u}_{tt} = -\omega^2 c^2 \hat{u}}_{\text{ODE in } \hat{u}}$

2) $\hat{u}(\omega,t) = A(\omega)\cos(\omega tc) + B(\omega)\sin(\omega tc)$

$\hat{u}(\omega,0) = \int_{\mathbb{R}} \underbrace{u(x,0)}_{f(x)}\,e^{-i\omega x}\,dx = \mathcal{F}[f] = \hat{f}(\omega)$

$\hat{u}_t(\omega,0) = \int_{\mathbb{R}} \underbrace{u_t(x,0)}_{g(x)}\,e^{-i\omega x}\,dx = \mathcal{F}[g] = \hat{g}(\omega)$

$\hat{u}(\omega,t) = A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t)$

$(= C(\omega)e^{ic\omega t} + D(\omega)e^{-ic\omega t}$

→ you will have $e^{i\omega x}$ in the integral $\mathcal{F}^{-1}$ ...

$$t = 0: \quad \hat{u}(\omega, 0) = C(\omega) + D(\omega) = \hat{f}(\omega)$$

$$\hat{u}_t(\omega, t) = ic\omega\left[C(\omega)e^{ic\omega t} - D(\omega)e^{-ic\omega t}\right]$$

$$\hat{u}_t(\omega, 0) = ic\omega\left[C(\omega) - D(\omega)\right] = \hat{g}(\omega)$$

$$= ic\omega\left[\hat{f}(\omega) - 2D(\omega)\right] = \hat{g}(\omega)$$

$$\rightarrow \quad ic\omega\,\hat{f}(\omega) - \hat{g}(\omega) = 2ic\omega\, D(\omega)$$

$$D(\omega) = \left(\tfrac{1}{2}\right)\hat{f}(\omega) - \frac{1}{2ic\omega}\hat{g}(\omega)$$

$$C(\omega) = \hat{f}(\omega) - D(\omega) = \left(\tfrac{1}{2}\right)\hat{f}(\omega) + \frac{1}{2ic\omega}\hat{g}(\omega)$$

3) Take $\mathcal{F}^{-1}$

$$\mathcal{F}^{-1}\left[C(\omega)e^{ic\omega t} + D(\omega)e^{-ic\omega t}\right](x, t)$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \underbrace{\left(\left(\tfrac{1}{2}\right)\hat{f}(\omega) + \frac{1}{2ic\omega}\hat{g}(\omega)\right)}_{C(\omega)} e^{ic\omega t} e^{i\omega x}\, d\omega \quad \ldots$$

$$\ldots + \frac{1}{2\pi}\int_{\mathbb{R}} \underbrace{\left[\left(\tfrac{1}{2}\right)\hat{f}(\omega) - \frac{1}{2ic\omega}\hat{g}(\omega)\right]}_{D(\omega)} e^{-ic\omega t} e^{i\omega x}\, d\omega$$

$$\rightarrow \quad \frac{1}{2\pi}\int_{\mathbb{R}} \left(\tfrac{1}{2}\right)\hat{f}(\omega)\, e^{i\omega(x+ct)}\, d\omega = \left(\tfrac{1}{2}\right) f(x+ct)$$

$$\left(= \frac{1}{2} \cdot \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\omega) e^{i\omega y}\, d\omega = \mathcal{F}^{-1}[\hat{f}](y) = f(y) \right)$$

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{2}\hat{f}(\omega)\, e^{i\omega(x-ct)}\, d\omega$$

$$= \left(\tfrac{1}{2}\right)\left(\frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\omega) e^{i\omega y}\, d\omega\right) = \frac{1}{2}\mathcal{F}^{-1}[\hat{f}](y)$$

$$= \frac{1}{2} f(y) = \frac{1}{2} f(x-ct)$$

TBC...