MATH

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Feb. 25/19

Recap:

Laplace Transform : $f : [0, +\infty) \to \mathbb{R}$

$\mathcal{L}[f](s) = \int_0^{+\infty} e^{-st} f(t)\, dt$

- linear : $\mathcal{L}[f + cg] = \mathcal{L}[f] + c\,\mathcal{L}[g]$
- invertible : $\mathcal{L}^{-1}[\alpha[f]] = f$

  if $\mathcal{L}[f] = \mathcal{L}[g] \Rightarrow f = g$
- NOT multiplicative : $\mathcal{L}[fg] \neq \mathcal{L}[f] \cdot [g]$

Use $\mathcal{L}$ to solve PDEs :

$u_{tt} = k u_{xx} \quad (k = c^2 > 0)$

$u(0, t) = u(1, t) = 0$

$u(x, 0) = f(x) \;\}\; u_t(x, 0) = g(x) \;\}$ need these @ $t = 0$

1) Take $\mathcal{L}$ of both sides (5 %)
2) Do assumptions (~ 95 %)
3) Take $\mathcal{L}^{-1}$ (~ << 1 %)

- Can ONLY take $\mathcal{L}$ in time $t \in [0, +\infty]$

Note :

$u(0, t) = \int_0^{+\infty} u(0, t) e^{-st}\, dt = 0, \quad U = \mathcal{L}[u]$

$u(1, t) = \int_0^{+\infty} u(1, t) e^{-st}\, dt = 0$

Today :
- Example of $\mathcal{L}^{-1}$
- Example of PDE solved via $\mathcal{L}$ and error function

  $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz$

Ex: Find $\mathcal{L}^{-1}\left[\dfrac{1}{s^2+2s+a}\right]$ in terms of a

($a \in \mathbb{R}$ parameter)

Complete the square:

$$s^2+2s+a = \underbrace{s^2+2s+1}_{\to (s+1)^2} + \underbrace{a-1}_{\to\ >0,\ =0,\ <0}$$

(1) IF $a-1>0$: $b=\sqrt{a-1}>0$

$$\frac{1}{s^2+2s+a} = \frac{1}{b}\,\frac{b}{(s+1)^2+b^2}$$

$$= (1/b)\,\mathcal{L}\left[e^{-t}\sin bt\right] \quad \text{(line 19)}$$

From table of laplace transforms

$$\to \mathcal{L}^{-1}\left[\frac{1}{s^2+2s+a}\right] = \left(\frac{1}{b}\right) e^{-t}\sin bt$$

$$= \frac{e^{-t}\sin\left(t\sqrt{a-1}\right)}{\sqrt{a-1}}$$

(2) IF $a-1=0$:

$$\frac{1}{s^2+2s+a} = \frac{1}{(s+1)^2} = \mathcal{L}\left[te^{-t}\right] \quad \text{(line 23)}$$

$$\to \mathcal{L}^{-1}\left[\frac{1}{s^2+2s+a}\right] = te^{-t}$$

(3) IF $a-1<0$: $b=\sqrt{1-a}>0$

$$\frac{1}{s^2+2s+a} = \frac{b}{(s+1)^2-b^2}\cdot\frac{1}{b} = \frac{1}{b}\,\mathcal{L}\left[e^{-t}\sinh bt\right] \quad \text{(line 21)}$$

$$\to \mathcal{L}^{-1}\left[\frac{1}{s^2+2s+a}\right] = \frac{1}{b}\,e^{-t}\sinh bt$$

$$= \frac{e^{-t}\sinh\left(t\sqrt{1-a}\right)}{\sqrt{1-a}}$$

Ex: Find $\mathcal{L}^{-1}\left[\frac{1}{s(s^2+s+1)}\right]$

Partial Fractions:

$$\frac{1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1} \rightsquigarrow \text{Solve } A, B, C$$

$$= \frac{As^2+As+A+Bs^2+Cs}{s(s^2+s+1)}$$

$\rightarrow$ $1 = s^2(A+B) + s(A+C) + A$

$A=1$, $0 = A+C \rightarrow C=-1$

$0 = A+B \rightarrow B=-1$

$$\rightarrow \frac{1}{s(s^2+s+1)} = \underbrace{\frac{1}{s}}_{=\mathcal{L}[1]\ (\text{line } 1)} - \frac{s+1}{s^2+s+1}$$

$$s^2+s+1 = \underbrace{s^2+s+1/4}_{(s+1/2)^2} + \underbrace{3/4}_{(\sqrt{3}/2)^2}$$

$$\frac{s+1}{(s+1/2)^2+(\sqrt{3}/2)^2} = \underbrace{\frac{s+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2}}_{\mathcal{L}[e^{-t/2}\cos((\sqrt{3}/2)t)]\ (\text{line } 20)} = \underbrace{\frac{1}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2}}_{\mathcal{L}[e^{-t/2}\sin((\sqrt{3}/2)t)]}$$

$$\rightarrow \mathcal{L}^{-1}\left[\frac{1}{s(s^2+s+1)}\right] = 1 - e^{-t/2}\cos((\sqrt{3}/2)t) - \frac{1}{\sqrt{3}}e^{-t/2}\sin((\sqrt{3}/2)t)$$

Ex: Solve $u_t = u_{xx}$

$u(0,t) = 0$ & $u(1,t) = 1$

$u(x,0) = 0$

Using $\mathcal{L}$

(1) Take $\mathcal{L}$: $\mathcal{L}[u_t](x,s) = sU(x,s) - \cancel{u(x,0)}$

$U = \mathcal{L}[u]$ $\quad$ $\mathcal{L}[u_{xx}](x,s) = U_{xx}(x,s)$

$\rightarrow$ $sU(x,s) = U_{xx}(x,s)$ $\quad$ $(s>0)$

ODE in U

$\rightarrow$ $U(x,s) = ae^{\sqrt{s}x} + be^{-\sqrt{s}x}$

• $U(0,s) = \int_0^{+\infty} u(0,t)e^{-st}\,dt = 0$

$$= ae^{\sqrt{s}\cdot 0} + be^{-\sqrt{s}\cdot 0} = a+b$$

$$U(1,s) = \int_0^{+\infty} u(1,t)e^{-st}\,dt = \frac{e^{-st}}{-s}\Big|_0^{+\infty} = \frac{1}{s}$$

$$= ae^{\sqrt{s}} + be^{-\sqrt{s}} = ae^{\sqrt{s}} \underbrace{-\, ae^{-\sqrt{s}}}_{b=-a}$$

$$\rightarrow \quad a = -b = \frac{1}{s(e^{\sqrt{s}} - e^{-\sqrt{s}})}$$

$$\rightarrow \quad U(x,s) = \frac{e^{\sqrt{s}x} - e^{-\sqrt{s}x}}{s(e^{\sqrt{s}} - e^{-\sqrt{s}})}$$

$$= \frac{\cancel{e^{\sqrt{s}}}\left(e^{\sqrt{s}(x-1)} - e^{-\sqrt{s}(x+1)}\right)}{\cancel{e^{\sqrt{s}}}\cdot s(1-e^{-2\sqrt{s}})}$$

$$= \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \longrightarrow \sum_{n=0}^{\infty}\left(e^{-2\sqrt{s}}\right)^n = \frac{1}{1-e^{-2\sqrt{s}}}$$

$$|x| < 1 \qquad x = e^{-2\sqrt{s}}$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{e^{-\sqrt{s}(2n+1-x)}}{s}}_{=\mathcal{L}\left[1-\operatorname{erf}\left(\frac{2n+1-x}{2\sqrt{t}}\right)\right]} + \underbrace{\sum_{n=0}^{\infty} \frac{-e^{-\sqrt{s}(2n+1-x)}}{s}}_{\mathcal{L}\left[1-\operatorname{erf}\left(\frac{2n+1+x}{2\sqrt{t}}\right)\right]}$$

$$\rightarrow \quad u(x,t) = \sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2n+1+x}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{2n+1-x}{2\sqrt{t}}\right)\right]$$

(where $\operatorname{erf}(x) = (2/\sqrt{\pi})\int_0^x e^{-z^2}\,dz$)

Taste of difficulty:

$$\mathcal{L}^{-1}\left[e^{-\sqrt{s}}\right] = \frac{1}{2\sqrt{\pi}}\, t^{-3/2} e^{-\frac{1}{4t}}$$

(requires quite a bit of complex integrals...)