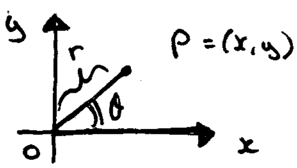
- Assignment #2 posted online (D2L)
  (due 11:59 pm on March 1st)
  ↪ 5 exercises (3 points each)

Recap:

- polar coords (2D)

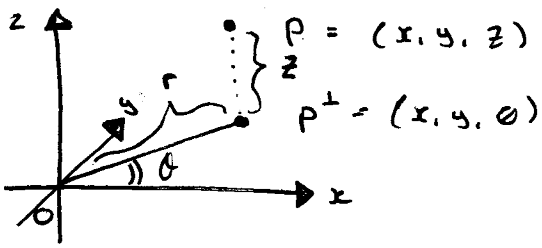


$r = \sqrt{x^2+y^2}$ $\qquad \theta = \arctan(y/x)$

$x = r\cos\theta$ $\qquad [0, 2\pi)$

$y = r\sin\theta$

- Cylindrical coords (3D)



$r = \sqrt{x^2+y^2}$

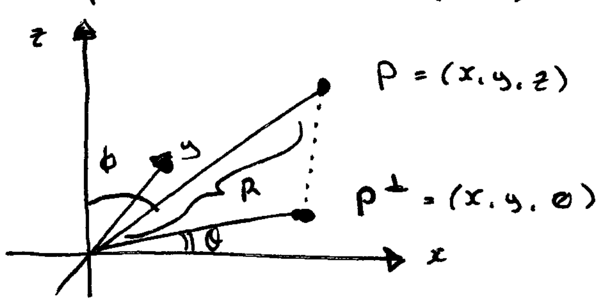
$\theta = \arctan(y/x)$

$z = z$

$x = r\cos\theta$

$y = r\sin\theta$

- Spherical coords (3D)



$R = \sqrt{x^2+y^2+z^2}$

$\theta = \arctan(y/x)$

$\phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

$x = R\cos\theta\sin\phi$

$y = R\sin\theta\sin\phi$

$z = R\cos\phi$

Today: Laplace eq. in cyl. / sph. coordinates

$u_{xx} + u_{yy} + u_{zz} = 0$

$= \Delta u = \nabla^2 u \qquad \Delta = \nabla^2$ (laplace operator)

Cylindrical coordinates:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Separation of variables: $u = R(r)\,\Theta(\theta)\,Z(z)$
↪ 2π periodic

Plug into PDE, do computations...

$$\frac{1}{r}\,\frac{R'(r)+rR''(r)}{R(r)} + \underbrace{\frac{1}{r^2}\,\frac{\Theta''(\theta)}{\Theta(\theta)}}_{=-m^2,\ m\in\mathbb{N}} + \underbrace{\frac{Z''(z)}{Z(z)}}_{=k^2>0} = 0$$

$\Theta(\theta) = a_1 \cos m\theta + b_1 \sin m\theta$

$Z(z) = a_2 e^{kz} + b_2 e^{-kz}$

For $R(r)$:

$$\left(\frac{1}{r}\right)\frac{R'(r)+rR''(r)}{R(r)} - \frac{m^2}{r^2} + k^2 = 0$$

multiply by $r^2 R(r)$:

$r^2 R''(r) + rR'(r) - m^2 R(r) + k^2 r^2 R(r) = 0$

Bessel eq.

$\rightarrow R(r) = a_3 \underline{\underline{J_m}}(kr) + b_3 \underline{Y_m}(kr)$

m^{th} Bessel Functions of $1^{ST}/2^{ND}$ kind

Solution:

$$u(r,\theta,z) = \sum_{k,m=0}^{\infty} \underbrace{\left[a_1\cos(m\theta) + b_1\sin(m\theta)\right]}_{\Theta(\theta)} \cdot (\cdots)$$

$$(\cdots) \cdot \underbrace{\left[a_2 e^{kz} + b_2 e^{-kz}\right]}_{Z(z)} \cdot (\cdots)$$

$$(\cdots) \cdot \underbrace{\left[a_3 J_m(kr) + b_3 Y_m(kr)\right]}_{R(r)}$$

Spherical coords:

$$\Delta u = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial u}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + (\cdots)$$

$$(\cdots) + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2} = 0$$

capital R', basically

Separation of Variables: $u = F(R)\underbrace{\Theta(\theta)}_{2\pi\text{ periodic}}\Phi(\phi)$

plug into PDE and do computations

$$\rightarrow \frac{1}{\Phi(\phi)\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\,\Phi'(\phi)\right) + \frac{1}{\Phi(\phi)^2\sin^2\phi}\frac{\Theta''(\theta)}{\Theta(\theta)} + (\cdots)$$

$$(\cdots) + \frac{1}{F(R)}\frac{\partial}{\partial R}\left(RF'(R)\right) = 0$$

$\Theta(\theta) = a_1 \cos m\theta + b_1 \sin m\theta$, $m \in \mathbb{N}$

$\mu = \cos\phi$, $M(\mu) = \Phi(\phi)$

$\rightarrow (1-\mu^2) M'(\mu) + (\lambda - \frac{m^2}{1-\mu^2}) M(\mu) = 0$

Associated Legendre equation

Solvable only when $\lambda = -\ell(\ell+1)$

For $\ell = \mathbb{N}$

and $-\ell \le m \le \ell$

$$M(\mu) = a_2 P_\ell^m(\mu) + b_2 Q_\ell^m(\mu)$$
$$= a_2 P_\ell^m(\cos\phi) + b_2 Q_\ell^m(\cos\phi)$$

Associated Legendre

Polynomials of $1^{ST}/2^{ND}$ type

For $F(R)$:

$$R^2 F''(R) + 2RF'(R) - \ell(\ell+1)F(R) = 0$$
$$\rightarrow F(R) = a_3 R^\ell + b_3 R^{-(\ell+1)}$$

Solution :

$$u(R,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[a_3 R^\ell + b_3 R^{-(\ell+1)} \right] Y_\ell^m(\theta,\phi)$$

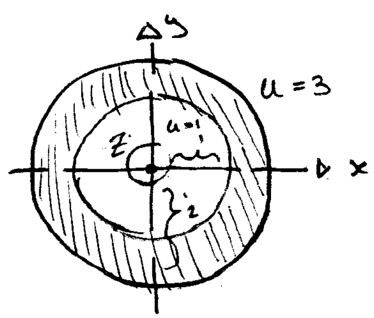
where

$$\underbrace{Y_\ell^m(\theta,\phi)}_{\text{Spherical harmonics}} = \underbrace{[a_1 \cos m\theta + b_1 \sin m\theta]}_{\Theta(\theta)} \cdot \underbrace{[a_2 P_\ell^m(\cos\phi) + b_2 Q_\ell^m(\cos\phi)]}_{\Phi(\phi)}$$

$\boxed{\text{Ex}}$ Spherical shell with unknown electrical field u :

- Inner shell : radius = 1 $u = 1$ here
- Outer shell : radius = 2 $u = 3$ here

Find u in the shell



$\Delta u = 0$

$u = 1$ on $\{R = 1\}$

$u = 3$ on $\{R = 2\}$

— use Spherical coords!

Pro-tip:

- Do boundary conditions depend on θ? NO $\to$ drop $\Theta(\theta)$
- Do " " depend on ϕ? NO $\to$ drop $\Phi(\phi)$
- Do " " depend on R? YES $\to$ keep $F(r)$

Why "NO $\to$ drop $\Theta(\theta)$"

$u = 1$ on $\{R = 1\}$:

$u(1, \theta, \phi) = F(1)\Theta(\theta)\Phi(\phi) = 1$

derive in θ:

$$\partial/\partial\theta \left[F(1)\Theta(\theta)\Phi(\phi) \right] = \partial/\partial\theta\, 1$$

$$= \underbrace{F(1)}_{\neq \emptyset}\underbrace{\Phi(\phi)}_{\neq \emptyset}\Theta'(\theta) = \emptyset \to \Theta(\theta) = \text{constant}$$

$u = F(R)$

$$\Delta u = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 F'(R)\right) + \cancel{\frac{1}{R^2 \sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)}^{\emptyset} + \cancel{\left(\frac{1}{R^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right)}^{\emptyset} = \emptyset$$

$$\to \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 F'(R)\right) = \emptyset$$

$$\to \frac{\partial}{\partial R}\left(R^2 F'(R)\right) = \emptyset \quad \lightning \quad R^2 F'(R) = a \text{ const.}$$

$$\to F'(R) = \frac{a}{R^2}, \quad F(R) = \frac{-a}{R} + b, \quad b \text{ const.}$$

$u = F(R) = 1$ on $\{R = 1\}$: $1 = F(1) = -a + b$

$u = F(R) = 3$ on $\{R = 2\}$: $3 = F(2) = \frac{-a}{2} + b$

$\to a/2 = 2, \quad a = 4$

$F(2) - F(1) = 2 \qquad b = 5$

$= (-a/2 + b) - (-a + b)$

Solution $u = F(R) = \dfrac{-4}{R} + 5$