MATH

1

Feb. 4/19

$u(L,t) = 0 \Rightarrow \sin hL = 0 \Rightarrow hL = n\pi$

$\Rightarrow X(x) = C_n \sin\left(\frac{n\pi x}{L}\right)$

$T(t) = a e^{\quad}$

Superposition:

$u(x,t) = \sum_{n=1}^{\infty} A_n e^{\left(-\frac{n\pi}{L}\right)^2 \kappa t} \sin\left(\frac{n\pi x}{L}\right)$

At $t = 0$: $f(x) = u(x,0)$

$= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$

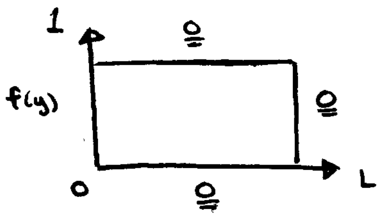
$A_n = \frac{1}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx} \int_L^0 f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

Today: Laplace eq'n $\Delta u = u_{xx} + u_{yy} = 0$

Wave eq'n $u_{tt} = c^2 u_{xx}$

with boundary conditions

$\boxed{\text{Example}}$



Electric Fied $u(x,y)$

$$= \begin{cases} 0 & \text{on } \{y=0\}\ \{x=L\} \\ & \{y=1\} \end{cases}$$

Find $u(x,y)$

$$\left.\begin{aligned} &u_{xx} + u_{yy} = 0 \quad \text{in } (0,L)\times(0,1) \\ &u(x,0) = u(x,1) = u(L,y) = 0 \\ &u(0,y) = f(y) \end{aligned}\right\}$$

1) First Solve PDE $u(x,y) = X(x)Y(y)$

$$\Rightarrow \underbrace{X'(x)Y(y)}_{u_{xx}} + \underbrace{X(x)Y'(y)}_{u_{yy}} = 0$$

$$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda \text{ const.}$$

(divide by $X(x)Y(y)$ here.)

$$\Rightarrow \left.\begin{aligned} X''(x) &= \lambda X(x) \\ Y''(y) &= -\lambda Y(y) \end{aligned}\right\}$$

$\Rightarrow$ use boundary conditions :

IF $\lambda = 0$: $X(x) = a_1 x + b_1$

$Y(y) = a_2 y + b_2$

$$u(x,0) = 0 = X(x)Y(0)_y = (a_1 x + b_1)(b_2) \longrightarrow b_2 = 0$$

$$u(x,1) = 0 = X(x)Y(1) = \underbrace{(a_1 x + b_1)}_{X(x)}\underbrace{a_2}_{Y(1)} \longrightarrow a_2 = 0$$

$a_2 = b_2 = 0 \longrightarrow Y(y) = 0$ (For all y)

$u(x,y) = X(x)Y(y) = 0$ (For all x)

→ You violate condition $u(0, y) = f(y)$ (no solution for $\lambda = 0$)

- IF $\lambda = h^2 > 0$

$X(x) = a_1 e^{hx} + b_1 e^{-hx}$

$Y(y) = a_2 \cos(hy) + b_2 \sin(hy)$

$0 = u(x, 0) = X(x)Y(0)$

$0 = Y(0) = a_2 \cosh 0 + b_2 \sinh 0 \rightarrow a_2 = 0$

$0 = u(x, 1) = X(x)Y(1)$

$\rightarrow 0 = Y(1) = b_2 \sinh \rightarrow h = n\pi, \; n \geq 1$

then $Y(y) = b_2 \sin(n\pi y)$

$0 = u(L, y) = X(L)Y(y)$

$\rightarrow 0 = X(L) = a_1 e^{hL} + b_1 e^{-hL}$

$\rightarrow b_1 = \dfrac{a_1 e^{hL}}{e^{-hL}} = -a e^{2hL}$

$f(y) = u(0, y)$ to use

(Careful: write generic solution first - most likely a series - before using Non-zero boundary conditions)

Generic Solution

$$u(x,y) = \sum_{n=1}^{\infty} \underbrace{(a_1 e^{n\pi x} + b_1 e^{-n\pi x})}_{X(x)} \cdot \underbrace{b_2 \sin(n\pi y)}_{Y(y)}$$

$$\Rightarrow \sum_{n=1}^{\infty} \underbrace{A_n}_{= a_1 b_2} \left[e^{n\pi x} - e^{2n\pi L} e^{-n\pi x} \right] \sin(n\pi y)$$

should get rid of "difficult" part.

Now we can use

$$f(y) = u(0, y) = \sum_{n=1}^{\infty} A_n \left[1 - e^{2n\pi L} \right] \sin(n\pi y)$$

→ Multiply by $\sin(n\pi y)$, then integrate over $y \in (0,1)$

$$\int_0^1 f(y)\sin(n\pi y)\,dy$$
$$= \sum_{n=1}^{\infty} A_n(1-e^{2n\pi L})\int_0^1 \sin(n\pi y)\sin(n\pi y)\,dy$$
$$= A_n(1-e^{2n\pi L})\int_0^1 \sin^2(n\pi y)\,dy$$
$$= 1/2$$
$$\rightarrow \quad A_m = \frac{2}{1-e^{2n\pi L}}\int_0^1 f(y)\sin(n\pi y)\,dy$$
$$u(x,y) = \sum_{n=1}^{\infty}\left\{\frac{2}{1-e^{2n\pi L}}\int_0^1 f(y)\sin(n\pi y)\,dy\right\}\left[e^{n\pi x} - e^{2n\pi L - n\pi x}\right]\sin(n\pi y)$$

- If $\lambda = h^2 < 0$

$$\left.\begin{aligned} X''(x) &= -h^2 X(x) \\ Y''(y) &= h^2 Y(y) \end{aligned}\right\}$$

$$X(x) = a_1\cosh hx + b_1\sinh hx$$
$$Y(y) = a_2 e^{hy} + b_2 e^{-hy}$$

$0 = u(x,0) = X(x)Y(0)$

$\rightarrow \quad 0 = Y(0) = a_2 + b_2 \rightarrow \underline{a_2 = -b_2}$ (*)

$0 = u(x,1) = X(x)Y(1)$

$\rightarrow \quad 0 = Y(1) = a_2 e^{h} + b_2 e^{-h} = a_2(e^{h} - e^{-h})$

If $a_2 = 0 \rightarrow Y(0) \Rightarrow u(x,y) = 0$

$\rightarrow$ violate $u(0,y) = f(y) \ldots$

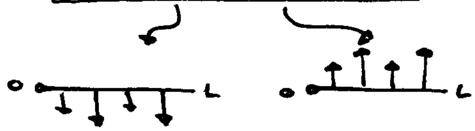
If $a_2 \neq 0 \rightarrow e^{h} - e^{-h} \ldots$ impossible

$\boxed{\text{No solution for } \lambda < 0}$

$\boxed{\textbf{Example}}$

oscillating string

$$u_{tt} = c^2 u_{xx}$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$



1) Solve PDE : $u(x,t) = X(x)T(t)$

$$\rightarrow \underbrace{X(x)T''(t)}_{u_{tt}} = c^2 \underbrace{X''(x)T(t)}_{u_{xx}}$$

$\rightarrow$ divide by $u = X(x)T(t)$:

$$\frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} = \lambda \text{ const.}$$

- If $\lambda = 0$

$$X(x) = a_1 x + b_1$$
$$T(t) = a_2 t + b_2$$

$a_2 = 0$ "non-oscillating"

$a_2 \neq 0$ then $a_2 t \cdot a_1 x \rightarrow \pm\infty$ as $t \rightarrow +\infty$ "infinite amplitude"

Physically, both $a_2 \neq 0$, $a_2 = 0$ are unrealistic ... expect no solution.

$$0 = u(0,t) = X(0)T(t)$$
$$\rightarrow 0 = X(0) = a_1 \cdot 0 + b_1 \rightarrow b_1 = 0$$
$$0 = u(L,t) = Y(L)T(t)$$
$$\rightarrow 0 = X(L) = a_1 L + b_1 \rightarrow a_1 = 0$$
$$\rightarrow X(x) = 0 \rightarrow u(x,t) = X(x)T(t) = 0$$

violating both $u(x,0) = f(x)$

$u_t(x,0) = g(x)$

$\boxed{\text{no solutions for } \lambda = 0}$

- can use physical intuition to guess when you have no solution ...
- But need math to prove no solutions ...

$$u_{tt} = c^2 u_{xx} \to \begin{aligned} u(0,t) &= u(1,t) = 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned}$$

$$u(x,t) = X(x)\,T(t)$$

$$c^2 X''(x)\,T(t) = T''(t)\,X(x)$$

$$\to \frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} = \lambda \text{ const.}$$

$$\lambda = 0 \quad \& \quad \lambda > 0 \quad \& \quad \lambda < 0$$

Case 1: $\lambda > 0$

$\lambda = h^2 > 0$

From 1 $\quad T''(t) = \lambda T(t)$

$$X''(x) = \frac{\lambda}{c^2} X(x) = \frac{\lambda^2}{c^2} X(x)$$

... ? TA trailed off.

Recap:

- Laplace eq'n $u_{xx} + u_{yy} = 0$
- Wave eq'n $u_{tt} = c^2 u_{xx}$, $c > 0$

1) $u = X(x)Y(y)$

$$\rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda \text{ constant}$$

2) $\lambda \leq 0$ no solutions

$\lambda > 0$: $X(x) = a_1 e^{hx} + b_1 e^{-hx}$

$Y(y) = a_2 \cosh y + b_2 \sinh y$ $\qquad h = n\pi$, $n = 1, 2, 3 \ldots$

$a_2 = 0$

$0 = u(L, y) \Rightarrow b_1 = -a_1 e^{2n\pi L}$

$$u(x,y) = \sum_{n=1}^{\infty} A_n \left[-e^{2n\pi L - n\pi x} + e^{n\pi x} \right] \cdot \sin(n\pi y)$$

To Find A_n: plug $x = 0$

$$f(y) = u(0,y) = \sum_{k=1}^{\infty} A_k \left[1 - e^{2k\pi L} \right] \sin(k\pi y)$$

mutiply by $\sin(n\pi y)$:

$$f(y) \sin(n\pi y) = \sum_{k=1}^{\infty} A_k \left[1 - e^{2k\pi L} \right] \sin(k\pi y) \sin(n\pi y)$$

Integrate over $y \in (0,1)$:

$$\int_0^1 f(y) \sin(n\pi y)\, dy$$

$$= \sum_{k=1}^{\infty} A_k \left[1 - e^{2k\pi L} \right] \overbrace{\int_0^1 \sin(k\pi y) \sin(n\pi y)\, dy}^{=0 \text{ if } k \neq n, \quad = 1/2 \text{ if } k = n}$$

$$= A_n \left[1 - e^{2\pi n L} \right] \left(\tfrac{1}{2} \right) = 0$$

$$\Rightarrow \int_0^1 f(y) \sin(n\pi y)\, dy = A_n \frac{1 - e^{2n\pi L}}{2}$$

$$\Rightarrow \frac{2}{1 - e^{2n\pi L}} \int_0^1 f(y) \sin(n\pi y)\, dy = A_n$$

- Wave eq'n : $u_{tt} = c^2 u_{xx}$

$u = X(x)T(t)$

$$\Rightarrow \frac{T''}{T} = c^2 \frac{X''}{X} = \lambda \text{ const.}$$

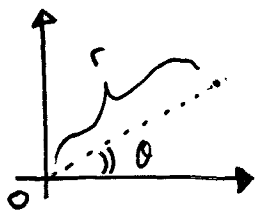
$\lambda = 0$: $T(t) = a_1 t + b_1$

$X(x) = a_2 t + b_2$

- $\{ a_1 = 0$ no oscillations $]$
- $\{ a_1 \neq 0,\ a_2 = 0$ always "flat" $\}$
- $\{ a_1 \neq 0,\ a_2 \neq 0$ oscillations get stronger in time $\}$

Today: polar, cylindrical, spherical coordinates

- Polar coordinates (2D)



$P = (x, y) = (r, \theta)$

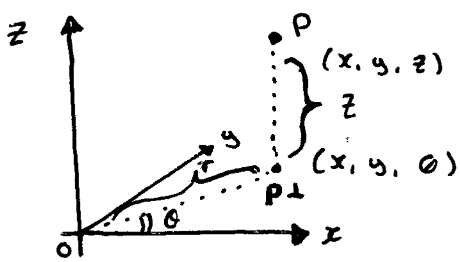
"radius" or "norm"   "anomaly" or "azimuth"

$r = \sqrt{x^2 + y^2} \quad (r \geq 0)$ $x = r\cos\theta$

$\theta = \arctan(y/x) \quad (0 \leq \theta \leq 2\pi)$ $y = r\sin\theta$

- useful with rotation symmetry
  (e.g. disk, annulus / ring, ...)

- Cylindrical coordinates (3D)



$= (r, \theta, z)$

$r = \sqrt{x^2+y^2}$

$\theta = \arctan(y/x)$

$z = z$

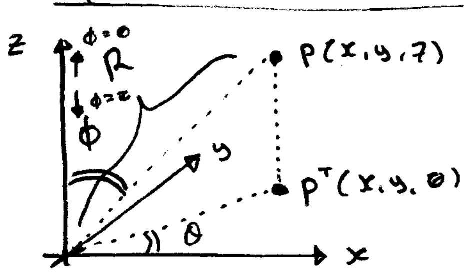
$x = r\cos\theta$

$y = r\sin\theta$

$P^{\perp}$ is projection of P on x-y plane

- useful with rotational symmetry around z-axis
  (e.g. cylinders, pipes, ...)

- Spherical coordinates (3D)



$P(x,y,z) = (R, \theta, \phi)$

$R = \sqrt{x^2+y^2+z^2}$

$\theta = \arctan(y/x)$

$\phi = \arctan(z/R)$

$= \arccos \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

$R \geq 0$

$0 \leq \theta < 2\pi$

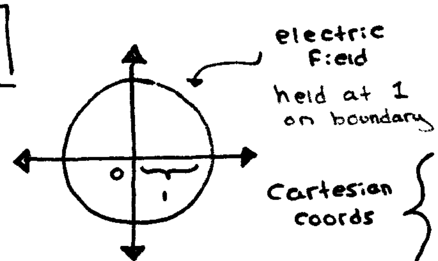
$0 \leq \phi \leq \pi$

Useful For spheres (or parts of sphere)

↳ bad For everything else

$$\begin{cases} x = R\cos\theta\sin\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\phi \end{cases} \qquad dx\,dy\,dz = \\ = R^2 \sin\phi\, dR\, d\theta\, d\phi$$

$\boxed{\text{Ex}}$



Find electric Field $\underline{u}$ in the disk.

Cartesian coords $\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,y) = 1 \\ \text{on } x^2+y^2 = 1 \end{cases}$

$$\downarrow$$

$$u(x, \pm\sqrt{1-x^2}) = 1$$

$$\boxed{\begin{array}{l} u_{xx} + u_{yy} = 0 \\ u(1,\theta) = 1 \\ \text{for all } \theta \in [0, 2\pi) \end{array}}$$

Polar cords

Suggest to drop θ

($\rightarrow$ 1 coordinate instead of 2 ...)

- Guess $u(r,\theta) = R(r)$ ⟵ "drop θ"

Need to write $u_{xx} + u_{yy} = 0$ in polar

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{(From books)}$$

How to do partial derivatives in polar:

$$\partial u/\partial x = (\partial u/\partial r)(\partial r/\partial x) + (\partial u/\partial\theta)(\partial\theta/\partial x)$$

$$r = \sqrt{x^2+y^2}$$

$$\rightarrow \frac{\partial r}{\partial x} = \frac{\partial x}{\partial\sqrt{x^2+y^2}} = \frac{\cancel{r}\cos\theta}{\cancel{r}} = \cos\theta$$

$$\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}(\arctan y/x) = \frac{1}{1+(y/x)^2}\cdot\left(-\frac{y}{x^2}\right)$$

$$= \frac{x^2}{x^2+y}\cdot(-y/x^2) = \frac{-y}{x^2+y^2}$$

$$= \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r} \longrightarrow \frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}$$

We solve: $\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\partial u/\partial r\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}} = 0$

$\Rightarrow \frac{1}{r}\left[\partial u/\partial r + r\,\partial^2 u/\partial r^2\right] = 0$ Since we guessed $u = R(r)$

Plug $u = R(r)$

$\Rightarrow \frac{1}{r}\left[R'(r) + r R''(r)\right] = 0$

Simplified:

$$R''(r) = \frac{-1}{r}R'(r) \longrightarrow \frac{R''(r)}{R(r)} = \frac{-1}{r}$$

$$\frac{R''(r)}{R'(r)} = \frac{d}{dr}\ln|R'(r)| \Rightarrow \frac{d}{dr}\ln|R'(r)| = -\frac{1}{r}$$

Integrating both sides:

$$\ln|R'(r)| = \int -\frac{1}{r}\,dr = -\ln r + c$$

Take exponential:

$$e^{\ln|R'(r)|} = |R'(r)| = e^{-\ln r + c} = e^c \cdot \frac{1}{r}$$

$$R'(r) = B \cdot \frac{1}{r}$$

$$B = \pm e^c$$

Integrate again:

$$R(r) = \int \underbrace{B \cdot \frac{1}{r}}_{R'(r)}\,dr = \underbrace{B\ln r + A}_{\int B \cdot \frac{1}{r}\,dr}$$

To find A:

$$1 = u(1,\theta) = R(1) = B\cancelto{0}{\ln(1)} + A \longrightarrow A = 1$$

How to find B:

- Task is to find u on <u>all</u> of the disk

But if $B \neq 0$: $R(r) = \underbrace{B\ln x + 1}_{\text{undefined}} = u(r,\theta)$

$(\rightarrow \pm\infty)$ at $r = 0$

not acceptable

$\longrightarrow$ need $B = 0$

Solution $u(r,\theta) = 1$

If we used cartesian: $u = X(x)Y(y)$

$\rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$ const.

- case $h = \sqrt{\lambda}$, $h > 0$ $(\lambda = h^2)$

$X(x) = a_1 e^{hx} + b_1 e^{-hx}$

$Y(y) = a_2 \cos hy + b_2 \sin hy$

$u(x, \pm\sqrt{1-x^2}) = 1 = X(x)Y(y) = X(x)Y(\sqrt{1-x^2})$

$\rightarrow (a_1 e^{hx} + b_1 e^{-hx})(a_2 \cos[\pm h\sqrt{1-x^2}] + b_2 \sin[\pm h\sqrt{1-x^2}] = 1$

— very lengthy to solve...