- ASSIGNMENTS GRADED IN 3-4 DAYS
- Intro to PDE

Arise From applications

$u = u(x,t)$

$u_t =$

Assume / Guess :

Today : Separable Variable PDE

Separable Variable PDE

From ODE : $y' = g(x)h(y)$

Divide by $h(y) \Rightarrow \frac{y'}{h(y)} = g(x)$

Integrate in x : $\int \frac{y'dx}{h(y)} = \int \frac{g(x)}{h(y)}$

$= g(x)\,dx$

if you can do the integrals, you get $h(y)$

For PDEs is similar :

1) Assume / Guess solution

$u(x,y) = X(x)Y(y)$

$u(x,y,z) = X(x)Y(y)Z(z)$

2) Plug back into PDE and divide by $u(x,y) = X(x)Y(y)$

3) You'll end up with : $\frac{X''(x)}{X(x)} = \frac{Y'(y)}{Y(y)} = \lambda$: const.

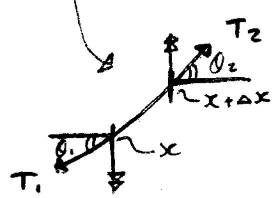
or something similar. (depending on PDE)

4) Now you have ODE's for $X(x)Y(y)$
↳ NOT PDEs

This works if the PDE is separable variable
(whether the PDE is separable variable or not, hard to see)

$\boxed{\text{Ex}}$ 
(string)
Let it vibrate

- Write the equation for its vertical displacement.



- T_1, T_2 tensions
- θ_1, θ_2 angle to horizontal axis

main quantity:

$u(x,t)$ = vertical displacement

$T_2 \sin\theta_2 = T_1 \sin\theta_1$ — but, $|T_1| = |T_2|$

$T(\sin\theta_2 - \sin\theta_1)$ = " vertical force on the string "

Force = mass × acceleration

$$\underbrace{T(\sin\theta_2 - \sin\theta_1)}_{\text{Force}} = \underbrace{\mu\,\Delta x}_{\text{mass}}\ \underbrace{u_{tt}(x,t)}_{\text{acceleration}}$$

$\theta_2 - \theta_1 \ll 1$

$$\Rightarrow T(\sin\theta_2 - \sin\theta_1) \cong T(\tan\theta_2 - \tan\theta_1)$$
$$= u_x(x+\Delta x, t) - u_x(x,t)$$
$$= u_x(x\,t) = \frac{1}{K}\,\frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}$$
$$\downarrow (\Delta x \to 0)$$
$$u_{xx}(x,t)$$

$$\Rightarrow \underbrace{u_{tt} = c^2 u_{xx} \qquad c^2 = 1/K > 0}_{\text{wave equation}}$$

$\boxed{\text{Ex.}}$ Solve $u_{tt} = c^2 u_{xx}$

Try separable variable method :

1) Assume $u(x,t) = X(x)T(t)$

2) Computations :

$$u_{tt} = X(x)T''(t) \qquad u_{xx} = X''(x)T(t)$$

Plug back into PDE :

$$\underbrace{X(x)T''(t)}_{u_{tt}} = c^2 \underbrace{X''(x)T(t)}_{u_{xx}}$$

3) Divide by $u = X(x)T(t)$

$$\frac{\cancel{X(x)}T''(t)}{\cancel{X(x)}T(t)} = \frac{X''(x)\cancel{T(t)}}{X(x)\cancel{T(t)}} c^2$$

$$\rightarrow \frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} = \lambda \quad \text{const.}$$

taking $\frac{\partial}{\partial t}$ gives

$$\partial/\partial t \left(\frac{T''(t)}{T(t)}\right) = c^2 \, \partial/\partial t\left(\frac{X''(x)}{X(x)}\right) = 0$$

4) We have ODE's

$$\left.\begin{aligned} T''(t) &= \lambda T(t) \\ X''(x) &= (\lambda/c^2) X(x) \end{aligned}\right\}$$

$\rightarrow$ i) If $\lambda = \underline{\underline{k^2 > 0}} \rightarrow \left.\begin{aligned} T''(t) &= k^2 T(t) \\ X''(x) &= (k/c)^2 X(x) \end{aligned}\right\}$

then $\begin{cases} T(t) = a_1 e^{kt} + b_1 e^{-kt} \\ X(x) = a_2 e^{(k/c)t} + b_2 e^{-(k/c)t} \end{cases}$

$\rightarrow$ ii) If $\lambda = \underline{\underline{k^2 < 0}} \longrightarrow \left.\begin{aligned} T''(t) &= k^2 T(t) \\ X''(x) &= (k/c)^2 X(x) \end{aligned}\right\}$

then $\begin{cases} T(t) = a_1 \cos(kt) + b_1 \sin(kt) \\ X(x) = a_2 \cos(\frac{k}{c} \cdot x) + b_2 \sin(\frac{k}{c} \cdot x) \end{cases}$

$\rightarrow$ iii) If $\lambda = 0 \longrightarrow T''(t) = X''(x) = 0$

then $\begin{cases} T(t) = a_1 t + b_1 \\ X(x) = a_2 x + b_2 \end{cases}$

The solution is $u(x,t) = X(x)T(t)$ in all 3 cases

$\boxed{\text{Ex}}$ Solve $u_x + u_y = u$

1) Assume $u(x,y) = X(x)Y(y)$

2) $u_x = X'(x)Y(y)$ & $u_y = X(x)Y'(y)$

Plug into PDE.

$$\underbrace{X'(x)Y(y)}_{u_x} + \underbrace{X(x)Y'(y)}_{u_y} = \underbrace{X(x)Y(y)}_{u}$$

3) Divide by $u = X(x)Y(y)$:

$$\frac{X'(x)}{X(x)} + \frac{Y'(y)}{Y(y)} = 1$$

$$\frac{X'(x)}{X(x)} = 1 - \frac{Y'(y)}{Y(y)} = \lambda \text{ const.}$$

4) You have ODEs:

$$X'(x) = \lambda X(x) \Rightarrow X(x) = ae^{\lambda x}$$
$$Y'(y) = (1-\lambda)Y(y) \quad Y(y) = be^{(1-\lambda)x}$$

Solution: $u(x,y) = X(x)Y(y) = ce^{\lambda x}e^{(1-\lambda)y}$

$\boxed{\text{Ex}}$ $yu_x + xu_y = 0$ (solve it)

1) Assume $u(x,y) = X(x)Y(y)$

2) $u_x = X'(x)Y(y)$ & $u_y = X(x)Y'(y)$

Plug into PDE

$$y\underbrace{(X'(x)Y(y))}_{u_x} + x\underbrace{(X(x)Y'(y))}_{u_y} = 0$$

3) Divide by $u = X(x)Y(y)$

$$y\left(\frac{X'(x)}{X(x)}\right) + x\left(\frac{Y'(y)}{Y(y)}\right) = 0$$

$$\Rightarrow y\left(\frac{X'(x)}{X(x)}\right) = -x\left(\frac{Y'(y)}{Y(y)}\right)$$

$$\left(\frac{1}{x}\right)\left(\frac{X'(x)}{X(x)}\right) = -\left(\frac{1}{y}\right)\left(\frac{Y'(y)}{Y(y)}\right) = \lambda \text{ constant}$$

4) You have ODEs

$$\left.\begin{aligned} X'(x) &= \lambda x X(x) \\ Y'(y) &= -\lambda y Y(Y) \end{aligned}\right\}$$

$$\frac{X'(x)}{X(x)} = \lambda x$$

Integrate: $\int \frac{X'(x)}{X(x)}\,dx = \ln|X(x)| = \frac{\lambda x^2}{2} + c$

Take exp:

$\exp(\ln|X(x)| = |X(x)| = e^{\frac{\lambda x^2}{2} + c}$

$\rightarrow \quad X(x) = ae^{\frac{\lambda x^2}{2} + c} \qquad a = \pm e^c$ Free const.

$\frac{Y'(y)}{Y(y)} = -\lambda y$ (same as before)

$Y(y) = be^{\frac{-\lambda y^2}{2}} \qquad b = \text{const.}$

Solution $u(x,y) = Ce^{\frac{\lambda x^2}{2}} e^{\frac{-\lambda y^2}{2}}$

1 $\partial u/\partial x = -3\,\partial u/\partial y$

$U(x,y) = X(x)Y(y)$

$$\frac{\partial u}{\partial x} = X'(x)\,Y(y) \qquad \frac{\partial u}{\partial y} = X(x)\,Y'(y)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)\,Y(y) \qquad \frac{\partial^2 u}{\partial y^2} = X(x)\,Y''(y)$$

$$\frac{X'(x)\cancel{Y(y)}}{X(x)\cancel{Y(y)}} = \frac{-3\cancel{X(x)}\,Y'(y)}{\cancel{X(x)}\,Y(y)}$$

$$\underbrace{\frac{X'(x)}{X(x)}}_{\text{Independent of } y} = -3\underbrace{\frac{Y'(y)}{Y(y)}}_{\text{Independent of } x} = \lambda \quad \text{const.} \quad (\text{Free parameter})$$

$$\frac{X'(x)}{X(x)} = \lambda \longrightarrow X'(x) = \lambda X(x)$$

$$X(x) = ae^{\lambda x} + be^{-\lambda x}$$

∘ ∘ ∘ his solution incorrect for $Y(y)$

2 $x(\partial u/\partial x) = y(\partial u/\partial y)$

$u(x,y) = X(x)Y(y)$

$$x\,\frac{X'(x)\cancel{Y(y)}}{X(x)\cancel{Y(y)}} = \frac{y\,\cancel{X(x)}\,Y'(y)}{\cancel{X(x)}\,Y(y)}$$

$$\underbrace{x\,\frac{X'(x)}{X(x)}}_{\text{Indep. of } y} = \underbrace{y\,\frac{Y'(y)}{Y(y)}}_{\text{Indep. of } x} = \lambda \quad \text{const.}$$

$$x^{-1}\left(x\,\frac{X'(x)}{X(x)} = \lambda\right)$$

$$\int\left[\frac{X'(x)}{X(x)} = \frac{\lambda}{x}\right]$$

$$\hookrightarrow \ln|X(x)| = \lambda \ln|x| + C$$

$$X(x) = ax^{\lambda}$$

∘ ∘ ∘ etc.

$\boxed{\text{Recap}}$

Separable Variable PDE :

1) Assume / Guess : $u = X(x)\,Y(y)$

$u = X(x)\,Y(y)\,Z(z)$

2) Do computations, plug into PDE
3) Divide by u; move all "x" one side, all "y" other side
4) Impose both sides = constant

$\rightarrow$ will get $\underbrace{\text{ODE}}_{\text{Not PDE's}}$ For $X(x)\,Y(y)$

Today : Separable variable PDE with boundary conditions.

- Superposition principle

Sum of Solutions of <u>linear</u> <u>homogeneous</u> PDE is again solution

$\rightarrow$ linear : linear in u and all its derivatives

$\boxed{\text{Ex.}}$ $u + x\,u_{yy} + \frac{1}{\sin y}\,u_x = 0$ linear

$\underline{u^2} + u_x = 0$ <u>NOT</u> Linear

$u + \underline{\frac{1}{u_x}} + y\,u_y + 2 = 0$ <u>NOT</u> Linear

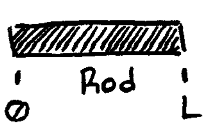
$\rightarrow$ homogeneous : <u>no</u> term has only x, y, or constants (so each term must be mutiplied by u or one of its derivatives)

$\boxed{\text{Ex.}}$ $\frac{1}{\tan x}\,e^{y}\,u_{xx} + x^2 u_y + y^3 u = 0$ homogeneous

$u_x + \underline{1} = 0$ <u>NOT</u> homogeneous

$u + u_{xx} + u_y + \underline{x} + \underline{y} = 0$ <u>NOT</u> homogeneous

$\boxed{\text{Ex.}}$



Heat the rod to temp. $f(x)$. Then let it cool. Forcefully keep both extremes at temperature $= 0$.

→ Find the temperature distribution $u(x,t)$

$u_t = k u_{xx}$

$u(x,\emptyset) = f(x)$ ← initial temperature when cooling starts

$\underbrace{u(\emptyset, L) = \emptyset = u(L,t)}$

Forcefully keeping extremes at temp $\emptyset$

1) Assume $u = X(x)\,T(t)$

2) $\underbrace{X(x)T'(t)}_{u_T} = k\underbrace{X''(x)T(t)}_{u_{xx}}$ $\quad k > \emptyset$

3) Divide by $u = X(x)T(t)$

$$\frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)} = \lambda \text{ const.}$$

4) $T'(t) = \lambda T(t)$
$X''(x) = (\lambda/k)X(x)$ $\Big\}$ $T(t) = ae^{\lambda t}, \quad a \in \mathbb{R}$

How "free" is a ?

$u(x,\emptyset) = f(x)$

$\Downarrow$

$X(x)T(\emptyset) = X(x)\cdot \overbrace{ae^{\lambda\cdot\emptyset}}^{T(\emptyset)} = aX(x)$ $\quad$ / take $t = \emptyset$

$\Rightarrow \quad a \neq \emptyset$

For $X''(x) = (\lambda/k)X(x)$:

→ 1) If $\lambda > \emptyset$: $h = \sqrt{\lambda/k} > \emptyset$

$\Rightarrow X(x) = be^{hx} + ce^{-hx}$ $\qquad \neq\emptyset$

$\emptyset = u(\emptyset,t) = X(\emptyset)T(t) = X(\emptyset)\cdot ae^{\lambda t}$

$\emptyset = X(\emptyset) = be^{h\cdot\emptyset} + ce^{-h\cdot\emptyset} = b + c \rightsquigarrow b = -c$

$\emptyset = u(L,t) = X(L)\underbrace{T(t)}_{\neq\emptyset}$

$\emptyset = X(L) = be^{hL} + ce^{-hL}$

$\Rightarrow \cancel{b}e^{hL} = -\cancel{c}e^{-hL} \rightarrow$ not possible

So no solution when $\lambda > \emptyset$

→ 2) IF $\lambda = 0$: $T(t) = a$

$X''(x) = 0 \rightarrow X(x) = bx + c$

$0 = u(0, t) = X(0) \cdot a$

$\Rightarrow X(0) = b \cdot 0 + c \rightarrow c = 0$

$0 = u(L, t) = X(L)T(t) = X(L) \cdot a$

$0 = X(L) = bL + \cancel{c}^{=0} \rightarrow b = 0$

$\Rightarrow X(x) = bx + c = 0$

$\Rightarrow u(x,t) = X(x)T(t) = 0$

(never possible due to $u(x, 0) = f(x)$
unless $f(x) = 0$ itself)

→ 3) IF $\lambda < 0$: $\left(\frac{\lambda}{K}\right) = -h^2 < 0 \rightarrow h = \sqrt{\frac{-\lambda}{K}} > 0$

$X''(x) = \frac{\lambda}{K} X(x)$

becomes $X''(x) = -h^2 X(x)$

$\Rightarrow X(x) = \underline{\underline{b}}\cos(hx) + \underline{\underline{c}}\sin(hx)$

↳ try to solve b, c

$0 = u(0, t) = X(0)T(t)$ $\qquad T(t) = ae^{\lambda t} \neq 0$

$\Rightarrow 0 = X(0) = b\cancel{\cos(h \cdot 0)}^{1} + \cancel{c\sin(h \cdot 0)}^{0}$

$\Rightarrow b = 0 \rightarrow X(x) = c\sin(hx)$

$0 = u(L, t) = X(L)T(t)$

$\Rightarrow 0 = X(L) = c\sin(hL)$

$\Rightarrow \sin(hL) = 0$

$\Rightarrow hL = n\pi, \quad n = 1, 2, 3 \ldots$

$h = \frac{n\pi}{L} \rightarrow \lambda = -h^2 K = \frac{-n^2\pi^2 K}{L^2}$

Solutions are :

$u(x, t) = X(x)T(t)$

with $X(x) = C_n \sin\left(\frac{n\pi x}{L}\right)$ $\qquad n = 1, 2, 3 \ldots$

$T(t) = ae^{\frac{-n^2\pi^2 K}{L^2}}$

Any

$$u_n = A_n e^{\frac{-n^2\pi^2 K}{L^2} t} \sin \frac{n\pi x}{L} \qquad n = 1, 2, 3 \ldots$$

is solution

$$\Rightarrow \underline{u = \sum_{n=1}^{\infty} A_n e^{\frac{-n^2\pi^2 k}{L^2} t} \cdot \sin\left(\frac{n\pi x}{L}\right)}$$

now we find A_n . . .

Use $u(x,0) = f(x)$:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \cdot \cancel{e^{\frac{-n^2\pi^2 k}{L^2}(0)}}^{\,1} \cdot \sin\left(\frac{\pi n x}{L}\right)$$

$$= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

To find A_m : multiply by $\sin\left(\frac{n\pi x}{L}\right)$

then integrate over $(0, L)$

$$\sum_{n=1}^{\infty} A_n \int_0^L \underbrace{\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx}_{*} = \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = \left(\frac{1}{2}\right)\left[\cos\left((n-m)\frac{\pi n x}{L}\right) - \cos\left((n+m)\frac{\pi n x}{L}\right)\right]$$

$$* = \frac{1}{2}\int_0^L \cos\left((n-m)\frac{\pi x}{L}\right) - \cos\left((n+m)\frac{\pi x}{L}\right) dx$$

if $n \neq m$ $= \left(\frac{1}{2}\right) \frac{\sin\left((n-m)\frac{\pi x}{L}\right)}{(n-m)(\pi/L)}\Bigg|_0^L - \left(\frac{1}{2}\right)\frac{\sin\left((n+m)\frac{\pi x}{L}\right)}{(n+m)(\pi/L)}\Bigg|_0^L = 0$

if $n = m$:

$$* = \int_0^L \sin^2\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}$$

$$A_m \cdot \left(\frac{L}{2}\right) = \int_0^L f(x) \sin\left(\frac{\pi n x}{L}\right) dx$$

LHS $= 0$ whenever $n \neq m$

$\rightarrow$ only term $n = m$ remains

$$\Rightarrow A_m = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

Solution :

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{2}{L}\int_0^L f(y) \sin\left(\frac{n\pi x}{L}\right) dy \right\} \cdot e^{\frac{-n^2\pi^2 k}{L^2} \cdot t} \cdot \sin\frac{n\pi x}{L}$$