MATH

1

JAN. 21/19

Bessel Functions

These are solution of Bessel equation

$[xy']' + (\alpha x^2 - \frac{n^2}{x})\, y = 0$

$A_2 J_n(\alpha b) + B_2 \alpha J_n'(\alpha b) = 0$

- $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$ parameters
- A_2, $B_2 \in \mathbb{R}$ not both 0
- domain $(0, b)$
- $J_n = y$ a solution of Bessel eq'n.

Bessel equations arise from electromagnetic (cylindrical laplace eq'n, spherical Helhmot eq'n)

- Bessel equation is a SL equation:

$[xy']' + (-\frac{n^2}{x} + \alpha^2 x)y = 0$ (Bessel)

$[r(x)]y']' + (q(x) + \lambda p(x))y = 0$ (SL)

$r(x), \; p(x) \geq 0$

$r(x) = x, \quad q(x) = \frac{-n^2}{2}, \quad p(x) = x, \quad \lambda = \alpha^2$

- Eigenvalues $\lambda_1 = \alpha_1^2 < \lambda_2 = \alpha_2^2 < \dots < \dots$
- Bessel eq'n has boundary condition ONLY at $x = b$, not at $x = 0$:

SL: has boundary condition at both a, b

$\Rightarrow$ each λ_n has 1 solution y_n

Bessel: boundary condition ONLY at $x = b$, nothing at $x = 0$.

$\Rightarrow$ each λ_n has $\underline{\underline{2}}$ solutions

$\underline{J_n(\alpha_n x)}$ ↳ Bessel function of first type

$\underline{Y_n(\alpha_n x)}$ ↳ Bessel function of second type

J_n and Y_n are <u>NOT</u> one multiple of the other.
(we will deal with J_n, much less with Y_n)

- Mutual Orthonality

$\int_0^b p(x) J_n(\alpha_n x) J_m(\alpha_m x)\, dx$

$= \int_0^b x J_n(\alpha_n x) J_m(\alpha_m x) = 0$ whenever $n \neq m$

Recurrence relations:

(1) $\frac{2\alpha}{x} J_\alpha(x) = J_{\alpha-1}(x) + J_{\alpha+1}(x)$ $\quad \alpha \in \mathbb{R}$

(2) $2J'_\alpha(x) = J_{\alpha+1}(x) - J_{\alpha+1}(x)$

Ex Knowing $J_{1/2}(x) = \sqrt{2/\pi x}\,\sin(x)$

$J_{-1/2}(x) = \sqrt{2/\pi x}\,\cos(x)$

Find: $J_{3/2}(x)$, $J_{-3/2}(x)$

→ Plug $\alpha = 1/2$ in (1):

$\frac{2\cdot 1/2}{x} J_{1/2}(x) = J_{-1/2}(x) + J_{3/2}(x)$

→ $J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$

$= \frac{1}{x}\sqrt{2/\pi x}\,\sin(x) - \sqrt{2/\pi x}\,\cos(x)$

You can get $J_{-3/2}(x)$ in two ways:

- Plug $\alpha = -1/2$ into (1):

$\frac{2\cdot(1/2)}{x} J_{-1/2}(x) = J_{-3/2}(x) + J_{1/2}(x)$

→ $J_{-3/2}(x) = \frac{1}{x} J_{-1/2}(x) - J_{1/2}(x)$

$= \sqrt{2/\pi x}\left(\frac{-\cos x}{x} - \sin x \right)$

- Plug $\alpha = -1/2$ in (2):

$2J'_{-1/2}(x) = J_{-3/2}(x) - J_{1/2}(x)$

→ $J_{-3/2}(x) = 2J'_{-1/2}(x) + J_{1/2}(x)$

$= 2\left[-\sqrt{\frac{2}{\pi x}}\sin x - \frac{1}{2}\sqrt{\frac{2}{\pi}}\, x^{-3/2}\cos x \right] + \sqrt{\frac{2}{\pi x}}\sin x$

$= -\sqrt{\frac{2}{\pi x}}\sin x - \frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x$

- Weighted squared norm:

$\|J_n\|^2 = \int_0^b x J_n{}^*(\alpha_n x)^2\, dx$

J_n itself depends on boundary conditions ⟹ so does $\|J_n\|^2$

$A_2 J_n(\alpha_n b) + B_2 J_n(\alpha_n b) = 0$

(boundary condition)

- $A_2 = 1$, $B_2 = 0$, $\|J_n(\alpha_i x)\|^2 = \frac{b^2}{2} J_{n+1}^2(\alpha_i b)$
- $A_2 = h \geq 0$, $B_2 = b$

$$\| J_n(\alpha_i x) \|^2 = \frac{\alpha_i^2 b - n^2 + h^2}{2\alpha_i^2} J_n(\alpha_i b)^2$$

- $A_2 = \emptyset$, $n = \emptyset$: $\| J_0(\alpha_i x)\|^2 = \frac{b^2}{2}$
- orthogonality :

$\int_0^b x J_n(\alpha_i x) J_m(\alpha_i x)\, dx = \emptyset$

$\int_0^b x J_n(\alpha_i x) J_n(\alpha_j x)\, dx = \emptyset$

$\Rightarrow$ $\boxed{\text{For fixed } n,\ \{ J_n(\alpha_i x) : i = 1, 2, \dots \} \text{ is orthogonal set}}$

Given $f : (\emptyset, b) \longrightarrow \mathbb{R}$:

$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$

How to find C_j :

$$f(x) \cdot J_n(\alpha_j x) = \int_0^b x f(x) J_n(\alpha_j x)\, dx$$

$$= \sum_{i=1}^{\infty} \underbrace{c_i J_n(\alpha_i x) \cdot J_n(\alpha_j x)}_{= \emptyset \text{ whenever } i \neq j} = C_j \| J_n(\alpha_j x)\|^2 \quad \text{(by mutual orthogonality)}$$

$$\Rightarrow C_j = \frac{1}{\| J_n(\alpha_j x)\|^2} \int_0^b x f(x) J_n(\alpha_j x)\, dx$$

Convergence :

- Fourier - Bessel series $= f(x)$ if x continuity point
- " $= \frac{f(x-1) + f(x+1)}{2}$ if x is jump

$f(x^-) = \lim_{y \to x^-} f(y)$; $f(x^+) = \lim_{y \to x^+} f(y)$

<u>Legendre Polynomials</u>

Solutions of $[(1-x^2) y']' + n(n+1) y = \emptyset$

Subject to boundary conditions

$$\left. \begin{aligned} A_1 y(-1) + B_1 y'(-1) &= \emptyset \\ A_2 y(1) + B_2 y'(1) &= \emptyset \end{aligned} \right\} \text{domain } (-1, 1)$$

- For each n, you have only 1 solution

$P_n(x) = n^{th}$ Legendre polynomial

- Legendre eq. is a SL eq'n with
  $r(x) = 1 - x^2$, $\lambda = n(n+1)$, $q(x) = 0$, $p(x) = 1$
- Mutual Orthogonality

$\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0$ whenever $n \neq m$

- Norm Square : $\|P_n\|^2 = \frac{2}{2n+1}$

$\{P_n(x) : n = 0, 1, 2, \cdots\}$ is orthogonal set

$\rightarrow$ $f : (-1, 1) \rightarrow \mathbb{R}$ can be expanded as

$$\underbrace{f(x) = \sum_{n=0}^{\infty} c_n P_n(x)}_{\text{Fourier-Legendre series}} \quad ; \quad c_n = \frac{2n+1}{x} \int_{-1}^{1} f(x) P_n(x)\, dx$$

Example 3 - From Textbook

assuming $P_0, P_1, P_2, \dots$ given.

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 \le x \le 1 \end{cases}$$

$C_0, C_1, C_2, ---, C_n$

$$C_0 = \tfrac{1}{2}\int_{-1}^{1} f(x) P_0(x)\,dx$$
$$= (1/2)\int_{-1}^{1} f(x)\, 1\, dx$$
$$= \left(\tfrac{1}{2}\right)\int_0^1 1 \cdot 1 \, dx$$
$$= (1/2)\left[x \big|_0^1 \right]$$
$$\rightarrow C_0 = (1/2)(1-0) = 1/2 \neq 0$$

↳ First term

$$C_1 = \tfrac{3}{2}\int_{-1}^{1} f(x) P_1(x)$$
$$= \tfrac{3}{2}\int_{-1}^{1} f(x)\, x$$
$$= 3/2 \int_0^1 1 x \, dx$$
$$= 3/2 \int_0^1 x \, dx$$
$$= 3/2 \left[\frac{x^2}{2} \Big|_0^1 \right]$$
$$\rightarrow C_1 = 3/4$$

↳ second term

$$C_2 = 5/2 \int_{-1}^{1} 1\, P_2(x)\, dx$$
$$= 5/2 \int_0^1 \tfrac{1}{2}(3x^2 - 1)\, dx$$
$$= (5/2)(1/2)\int_0^1 (3x^2 - 1)\, dx$$
$$= (5/4)\left[\cancel{3}x^3/\cancel{3} - x\right]_0^1$$
$$= (5/4)\left[x^3 - x\right]_0^1$$
$$\rightarrow C_2 = 0$$

$$C_3 = (7/2)\int_{-1}^{1} f(x) P_3(x)\, dx$$
$$= (7/2)\int_0^1 1\left(1/2\,(5x^3 - 3x)\right) dx$$
$$= (7/2)(1/2)\int_0^1 (5x^3 - 3x)\, dx$$
$$= (7/4)\left[\frac{5x^4}{4} - \frac{3x^2}{2}\right]_0^1$$
$$= (7/4)\left[5/4 - 3/2\right] - \left[0 - 0\right]$$
$$\rightarrow C_3 = -7/16$$

↳ third term

$$C_4 = 9/2 \int_{-1}^{1} f(x) P_4(x)\, dx$$
$$= (9/2)\int_0^1 1 \cdot (1/8) \cdot (35x^4 - 30x^2 + 3)\, dx$$
$$= (9/16)\int_0^1 (35x^4 - 30x^2 + 3)\, dx$$
$$= (9/16)\left[\frac{35x^5}{5} - \frac{30x^3}{3} + 3x\right]_0^1$$
$$\rightarrow C_4 = 0$$

$$C_5 = \left(\tfrac{11}{2}\right)\int_{-1}^{1} f(x) P_5(x)$$
$$= \left(\tfrac{11}{2}\right)\int_0^1 (1)(1/8)(63x^5 - 70x^3 + 15x)\, dx$$
$$= \left(\tfrac{11}{16}\right)\left[\frac{63x^6}{6} - \frac{70x^4}{4} + \frac{15x^2}{2}\Big|_0^1\right]$$
$$= (11/16)\left[\left[\frac{63}{6} - \frac{70}{4} + \frac{15}{2}\right] - 0\right]$$
$$= (11/16)\left[\frac{126}{12} - \frac{210}{12} + \frac{90}{12}\right]$$
$$\rightarrow C_5 = 11/32$$

↳ Fourth term

Thus,

$$f(x) = \left(\tfrac{1}{2}\right)P_0(x) + (3/4)P_1(x) - \dots$$
$$\dots (7/16)P_3(x) + (11/32)P_5(x)$$

For reference,

$$C_6 = 0$$
$$C_7 = \frac{65}{256}$$

(But we only wanted the first 4 terms)

Recap:

- Bessel eq'n : $[xy']' + (\alpha^2 x - \frac{n^2}{x})y = \emptyset$
- Legendre eq'n : $[(1-x^2)y']' + n(n+1)y = \emptyset$
- Mutual orthogonality :
  - Bessel : $\int_0^b x J_n(\alpha_i x) J_n(\alpha_j x)\,dx = \emptyset$ whenever $i \neq j$
  - Legendre : $\int_{-1}^{1} P_n(x) P_m(x)\,dx = \emptyset$ whenever $n \neq m$
  - Fourier - Bessel series : $f(\emptyset, b) \longrightarrow \mathbb{R}$

    $\sum_{i=1}^{\infty} c_i J_n(\alpha_i x_i)$; $c_i = \frac{1}{\| J_n(\alpha_i x_i) \|^2} \int_0^b x f(x) J_n(\alpha_i x_i)\,dx$

    $= \int_0^b x J_n^2(\alpha_i x_i)\,dx$
  - Fourier- Legendre series : $f : (-1, 1) \longrightarrow \mathbb{R}$

    $\sum_{i=1}^{\infty} C_n P_n(x)$, $C_n = \frac{1}{\|P_n\|^2} \int_{-1}^{1} f(x) P_n(x)\,dx$

    $= \int_{-1}^{1} P_n(x)\,dx$

Today: Intro to Partial Differential Equations (PDEs)

Partial Derivative :

$\partial u/\partial x\,(x_0, y_0) = \lim_{h \to \emptyset} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h}$

$\partial u/\partial y\,(x_0, y_0) = \lim_{h \to \emptyset} \frac{u(x_0, y_0 + h) - u(x_0, y_0)}{h}$

PDEs are just equations involving some partial derivative

$$F(\underbrace{x, y}_{\text{spatial eq'n}} ; \underbrace{u, \partial u/\partial x, \partial u/\partial y}_{\text{partial derivative}}, \dots) = \emptyset \qquad (*)$$

- A solution to $(*)$ is any function $u(x, y)$ such that $(*)$ holds for all x, y

2nd order linear PDEs

$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} \dots$

$\dots + D(x,y)u_x + E(x,y)u_y + F(x,y)u \dots$

$\dots = R(x,y)$

$u_x = \partial u/\partial x$

$u_y = \partial u/\partial y$ $\circ \circ \circ$

$u_{xy} = \partial^2 u/\partial x \partial y$

- Assume: $D(x,y) = E(x,y) = F(x,y) = R(x,y) = \emptyset$
  $A, B, C \in \mathbb{R}$ constants instead of functions

$\Rightarrow \underline{Au_{xx} + Bu_{xy} + Cu_{yy} = \emptyset}$
↳ constant coeff. PDE

$\boxed{\text{Ex.}}$ Solve: $u_{xx} + u_{yy} = \emptyset$

"Guess" solution $u(x,y) = X(x)Y(y)$

$$u_{xx} = \partial^2/\partial x^2 \left[X(x)Y(y)\right] = X''(x)Y(y)$$

$$u_{yy} = \partial^2/\partial y^2 \left[X(x)Y(y)\right] = X(x)Y''(y)$$

$$\Rightarrow \underbrace{X''(x)Y(y)}_{u_{xx}} + \underbrace{X(x)Y''(y)}_{u_{yy}} = \emptyset$$

$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \emptyset \qquad \text{(divided by } X(x)Y(y)\text{)}$$

$$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = K$$

$$\Rightarrow \left.\begin{aligned} X''(x) &= KX(x) \\ Y''(y) &= -KY(y) \end{aligned}\right\}$$

1) If $K > \emptyset$: $X(x) = ae^{x\sqrt{K}} + be^{-x\sqrt{K}}$
$Y(y) = c\cos(y\sqrt{K}) + d\sin(y\sqrt{K})$

Solution:
$u(x,y) = X(x)Y(y) = (ae^{x\sqrt{x}} + be^{-x\sqrt{K}})(c\cos(y\sqrt{K} + d\sin(y\sqrt{K}))$

2) $K < \emptyset$: $X(x) = a\cos(x\sqrt{K}) + b\sin(x\sqrt{K})$
$Y(y) = ce^{y\sqrt{-K}} + de^{-y\sqrt{-K}}$

3) $K = \emptyset$: $X''(x) = \emptyset$, $Y''(y) = \emptyset$
$X(x) = ax + b$, $Y(y) = cy + d$
$u(x,y) = X(x)Y(y)$
$= (ax+b)(cy+d)$ solution

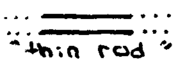
- Relax assumptions:
  $A, B, C, D, E, F \in \mathbb{R}$ constants
  $R(x,y) = \emptyset$

$$\Rightarrow A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = 0$$

$$\Delta = B^2 - 4AC$$

1) $\Delta > 0$: hyperbolic PDE
2) $\Delta < 0$: elliptic PDE
3) $\Delta = 0$: parabolic PDE

$u_{xx} + u_{yy} = 0$
had $A = C = 1$
$\Delta = -4$ elliptic

$\boxed{\text{Ex.}}$ 

Find equation for temperature

temperature → $u = u(x,t)$
at point x, time t

→ $\Delta x \ll 1$

Heat content : $\underset{\text{heat}}{Q} = C \cdot \underset{\text{temp.}}{u} \cdot \Delta x$ ← "mass"/length

Heat Flux : $Q_t = -k\left[u_x(x+\Delta x, t) - u_x(x,t)\right]$

Derive $Q = cu\Delta x$: $Q_t = c\Delta x\, u_t$
(in time)

$$\Rightarrow Q_t = c\Delta x\, u_t = -k\left[u(x+\Delta x, t) - u(x,t)\right]$$

$$u_t = -k/c \left(\frac{u_x(x+\Delta x, t) - u_x(x,t)}{\Delta x}\right)$$

$$\Delta x \to 0 \; : \; \underbrace{u_t = \frac{-k}{c} \cdot u_{xx}}_{\text{heat eq'n}}$$

$\left(\frac{k}{u}\right) u_{xx} + u_t = 0$ has the form
$A u_{xx} + B u_{xt} + C u + D u_x + E u_t + F u$
with $A = k/c$, $E = 1$, $B = C = D = F = 0$

✱ $\boxed{\text{Ex.}}$ Solve $u_t = -\left(\frac{k}{c}\right) \cdot u_{xx}$

"Guess" $u(x,t) = X(x)\,T(t)$

$u_t = X(x)\,T'(t)$, $u_{xx} = X''(x)\,T(t)$

$$\Rightarrow X(x)\,T'(t) = -\frac{k}{c} X''(x)\,T(t)$$

$$\Rightarrow \frac{T'(t)}{T(t)} = \frac{-k}{c} \frac{X''(x)}{X(x)} = \lambda$$

$$\Rightarrow \left.\begin{aligned} T'(t) &= \lambda T(t) \\ X''(x) &= \frac{-c}{k} \lambda X(x) \end{aligned}\right\} \text{ Heat eq'n is parabolic}$$

$T(t) = ae^{\lambda t}$

1) $\lambda > 0 \Rightarrow -\frac{c}{k}\lambda < 0 \rightarrow X(x) = b\cos\left(x\sqrt{\frac{c}{k}\lambda}\right) \cdots$

$\cdots + d\sin\left(x\sqrt{\frac{c}{k}\lambda}\right)$

2) $\lambda < 0 \Rightarrow -\frac{c}{k}\lambda > 0 \rightarrow X(x) = be^{x\sqrt{(c/k)\lambda}} + de^{-x\sqrt{(c/k)\lambda}}$

3) $\lambda = 0 \Rightarrow X''(x) = 0 \rightarrow X(x) = bx + d$

Solution is $u(x,t) = X(x)T(t)$

(different u for different λ ...)

$\boxed{\text{Ex.}}$ Try to solve $u_x + u_y = u$

Guess $u(x,y) = X(x)Y(y)$:

$$\underbrace{X'(x)Y(y)}_{u_x} + \underbrace{X(x)Y'(y)}_{u_y} = \underbrace{X(x)Y(y)}_{u}$$

divide by $X(x)Y(y)$:

$$\frac{X'(x)}{X(x)} + \frac{Y'(y)}{Y(y)} = 1 \longrightarrow \frac{X'(x)}{X(x)} = 1 - \frac{Y'(y)}{Y(y)} = k$$

$$\Rightarrow \left.\begin{aligned} X'(x) &= k\cdot X(x) \\ Y'(y) &= (1-k)Y(y) \end{aligned}\right\}$$

$X(x) = ae^{kx}, \quad Y(y) = be^{(1-k)y}$

Solution:

$$u(x,y) = \underbrace{ae^{kx}}_{X(x)} \cdot \underbrace{be^{(1-k)y}}_{Y(y)}$$