MATH

①

JAN. 14 /19

## Recap:

- Inner product:

  $f \cdot g = \int_D f(x) g(x) dx$

  $f \cdot g$ depends on domain D too

other symbol:

$\langle f, g \rangle$

$(f, g)$

- Orthogonality:

  $f \perp g \Leftrightarrow f \cdot g = 0$

- Orthogonal set:

  $\{ f_n : n = 1, 2, 3, \ldots, \}$ where

  $f_k \perp f_j$ whenever $k \neq j$

  If the only continuous function orthogonal to all $f_n$, $n = 1, 2, 3, 4, \ldots$ is the function $g(x) = 0 \Rightarrow$ complete orthogonal set.

- Norm: $\|f\| = \sqrt{f \cdot f}$

  $= \sqrt{\int_D f(x)^2 dx}$

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## Today: Fourier Series

Summation / Series symbol

$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \ldots$

### Fourier Series

$\{ \sin(nx) : n = 1, 2, 3, \ldots \}$ $\{ \cos(nx) : n = 1, 2, 3, \ldots \}$

both orthogonal sets

$\{ \sin(nx), \cos(nx), n = 1, 2, 3, \ldots$

$g(x) = 1 \}$ is still orthogonal set on $(-\pi, \pi)$

- More in general:

$\{1, \sin\frac{\pi n x}{p}, \cos\frac{\pi n x}{p} : n = 1, 2, 3, \dots\}$

is orthogonal set on $(-p, p)$, $p > 0$

↳ (will be practice problem with solution)

Fourier Series: $f : (-p, p) \to \mathbb{R}$

$$f(x) = \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{\pi n x}{p} + b_n \sin\frac{\pi n x}{p}\right]}_{\text{Fourier series of } f \text{ on domain } (-p, p)}$$

$a_0, a_n, b_n$ are Fourier coefficients

Take $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{\pi n x}{p} + b_n \sin\frac{\pi n x}{p}\right]$

and take inner product with $\cos\frac{\pi n x}{p}$:

$$f(x) \cdot \cos\frac{\pi n x}{p} = \left\{\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{\pi n x}{p} + b_n \sin\frac{\pi n x}{p}\right]\right\} \cdot \cos\frac{\pi n x}{p}$$

$$= a_n \left\| \cos\frac{\pi n x}{p} \right\|^2$$

$$\Rightarrow \quad a_n = \frac{1}{\left\| \cos\frac{\pi n x}{p} \right\|^2} \int_{-p}^{p} f(x) \cos\frac{\pi n x}{p}\, dx$$

$$\left\| \cos\frac{\pi n x}{p} \right\|^2 = \int_{-p}^{p} \cos^2\frac{\pi n x}{p}\, dx$$

$$= \frac{1}{2} \int_{-p}^{p} \left(1 + \cos\frac{2\pi n x}{p}\right) dx = p$$

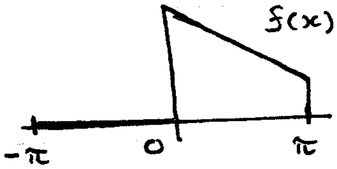
$$\Rightarrow \left.\begin{aligned} a_n &= \frac{1}{p} \int_{-p}^{p} f(x) \cos\frac{\pi n x}{p}\, dx \\ a_0 &= \frac{1}{p} \int_{-p}^{p} f(x)\, dx \\ b_n &= \frac{1}{p} \int_{-p}^{p} f(x) \sin\frac{\pi n x}{p}\, dx \end{aligned}\right\} \text{Fourier coefficients}$$

Note: $f(x) \cdot 1 = \int_{-p}^{p} f(x)\, dx = \int_{-p}^{p} \frac{a_0}{2}\, dx$

$$\Rightarrow a_0 = \frac{1}{p} \int_{-p}^{p} f(x)\, dx$$

$\Rightarrow$ the denominator "2" in $\frac{a_0}{2}$ is only to have $a_0 = a_n$ for $n = 0$

**Example** $f(x) = \begin{cases} 0 & \text{if } \ x \in (-\pi, 0) \\ \pi - x & \text{if } \ x \in (0, \pi) \end{cases}$



→ Find its Fourier Coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \overset{0 \text{ here}}{\cancel{\frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx}} = \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$\Rightarrow \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\,dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{\pi n x}{\pi}\,dx$$

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{0} f(x)\overset{0 \text{ here}}{\cancel{\cos(nx)}}\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,dx$$

$$\Rightarrow \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\cos(nx)\,dx$$

$$\Rightarrow -\frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx$$

$$\Rightarrow -\frac{1}{\pi}\left[x\overset{0}{\cancel{\frac{\sin(nx)}{n}}}\Big|_{0}^{\pi} - \int_{0}^{\pi}\frac{\sin(nx)}{n}\,dx\right] = \frac{1-(-1)^n}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \overset{0 \text{ here}}{\cancel{f(x)\sin(nx)}}\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{\pi\cos(nx)}{n}\Big|_{0}^{\pi} - \int_{0}^{\pi} x\sin(nx)\,dx\right]$$

$$= \frac{1-(-1)^n}{n} - \left[x - \frac{\cos(nx)}{n}\Big|_{0}^{\pi} + \int_{0}^{\pi}\overset{0}{\cancel{\frac{\cos(nx)}{n}}}\,dx\right]$$

always? $= \frac{1-(-1)^n}{n} + \frac{(-1)^n}{n} = \frac{1}{n} \quad ↯ \left[\cos(n\pi) = (-1)^n\right]$

$$f(x) \boxed{=} \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{\pi n x}{P} + b_n \sin\frac{\pi n x}{P}\right]$$

Previous example :

$$f(x) = \begin{cases} 0 & x \in (-\pi, 0) \\ \pi - x & x \in (0, \pi) \end{cases}$$

Fourier series : $\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$

$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2\pi}\cos(nx) + \frac{1}{n}\sin(nx)\right]$$

@ $x = 0$, $f(0) = \pi$ but Fourier series

$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2\pi} = \frac{\pi}{2} \neq f(0) \ldots$$

Convergence of Fourier Series:

assume $f \cdot f'$ are piece-wise continuous

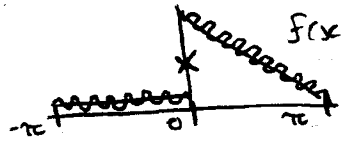
↳ has only finitely many jump discontinuities

then:

Fourier series = $f(x)$ at all continuity points $x$

Fourier series = $\dfrac{f(x_0^-) + f(x_0^+)}{2}$ at jumps $x_0$

$f(x_0^{\pm}) = \lim\limits_{y \to x_0^{\pm}} f(y)$



∿∿∿ = Fourier series

More names:

$$f(x) = \frac{a_0}{2} + \underbrace{\sum_{n=1}^{\infty} a_n \cos \frac{\pi n x}{P}}_{\text{Fourier cosine series}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin \frac{\pi n x}{P}}_{\text{Fourier sine series.}}$$

$f(x)$ = Fourier cosine series if

$f$ is even $(f(x) = f(-x))$

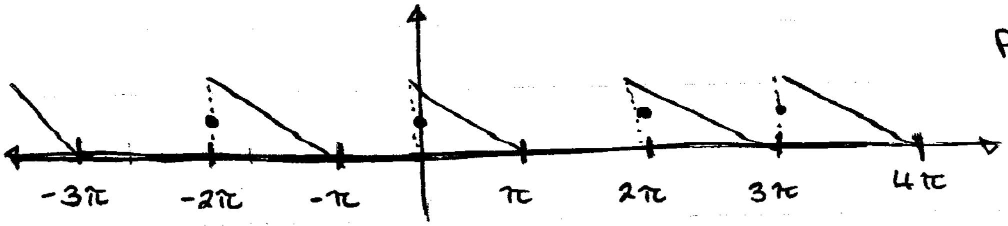
- $f(x)$ = Fourier sine series if

$f$ is odd $(f(x) = -f(-x))$

From previous example:

- $f$ is defined ONLY in $(-\pi, \pi)$
- Fourier series defined for all $x = \mathbb{R}$

⇒ Fourier series is Periodic extension of $f$ outside of its domain $(-\pi, \pi)$



Plot of Fourier series of $f$

complex Fourier Series
$f(x) = \sum^{+\infty} C_n e^{\frac{-n\pi x}{P}}$

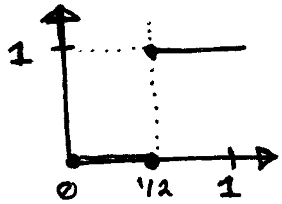
If $f$ given only as $f: [0, L] \to \mathbb{R}$:

you'll have 3 different eqns }
1) using Fourier cosine series ‖ "half range
2) " " sine series ‖ extensions "
3) using Fourier series

$$f(x) = \begin{cases} 0 & \text{if } x \in (0, 1/2) \\ 1 & \text{if } x \in [1/2, 1) \end{cases} \qquad f(0,1) \to \mathbb{R}$$

Extend by periodicity using

1) Fourier sine series
2) Fourier cosine series
3) Fourier series



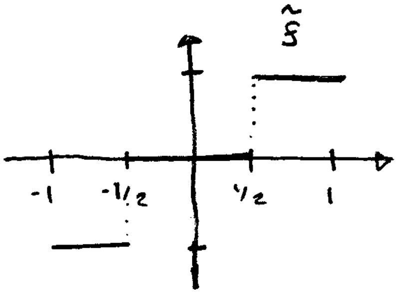
→ (1) Extended by Fourier sine series

- All 3 Fourier series <u>NEED</u> domain to be $(-p, p)$ For some $p > 0$ …
- Fourier sine series = Fourier series if function is <u>odd</u>

→ need first to extend $f(0,1) \to \mathbb{R}$ to some $\tilde{f} : (-1, 1) \to$ <u>ODD</u>

$$\tilde{f}(x) = \begin{cases} -1 & \text{if } x \in (-1, -1/2] \\ 0 & \text{if } x \in (-1/2, 1/2) \\ 1 & \text{if } x \in [1/2, 1) \end{cases}$$

↳ is only odd function such that $\tilde{f} = f$ on $(0, 1)$



Fourier sine series:

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{p} \qquad p = 1 \text{ here}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \tilde{f}(x) \sin\left(\frac{n\pi x}{p}\right) dx$$

$$= \int_{-1}^{1} \tilde{f}(x) \sin(n\pi x)\, dx$$

$\tilde{f} = 0$ here

$$\Rightarrow \int_{-1}^{-1/2} \tilde{f}(x) \sin(n\pi x)dx + \int_{-1/2}^{1/2} \cancel{\tilde{f}(x)} \sin(n\pi x)\, dx + \dots$$

$$\dots \int_{1/2}^{1} \tilde{f}(x) \sin(n\pi x)\, dx$$

$$\Rightarrow \int_{-1}^{-1/2} \sin(n\pi x)dx + \int_{1/2}^{1} \sin(n\pi x)dx$$

$$= \frac{\cos(n\pi x)}{n\pi}\Big|_{-1}^{-1/2} - \frac{\cos(n\pi x)}{n\pi}\Big|_{1/2}^{1}$$

$$= \frac{\cos\left(\frac{-n\pi}{2}\right) - \cos(n\pi)}{n\pi} - \frac{\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)}{n\pi}$$

$$= \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right)$$

= 0 if n odd

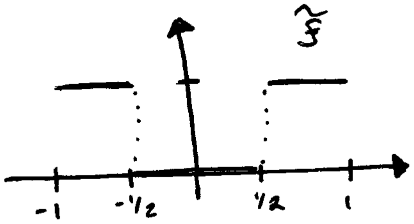
= 1 if n multiple of 4

-1 if n is even, but not multiple of 4

⇒ Extending by Fourier Sine Series gives $\sum_{n=1}^{\infty} \frac{2}{n\pi}\underbrace{\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right)}_{b_n} \sin(n\pi x)$

2) Extend by Fourier cosine series

Need to extend $f$ to $(-1,1)$ and have an <u>even</u> function.



$$\tilde{f}(x) = \begin{cases} 1 & \text{if } x \in (-1, -1/2] \\ 0 & \text{if } x \in (-1/2, 1/2) \\ -1 & \text{if } x \in [1/2, 1) \end{cases}$$

(where $p = 1$)

Fourier Cosine series: $\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right) + \frac{a_0}{2}$ & $a_n = \frac{1}{p}\int_{-p}^{p} \tilde{f}(x)\cos\left(\frac{n\pi x}{p}\right)dx$

$$a_0 = \int_{-1}^{1} \tilde{f}(x)\,dx$$
$$= \int_{-1}^{-1/2} 1\,dx + \cancelto{0}{\int_{-1/2}^{1/2} 0\,dx} + \int_{1/2}^{1} 1\,dx = 1$$

$$a_n = \int_{-1}^{1} \tilde{f}(x)\cos(n\pi x)\,dx$$
$$= \int_{-1}^{-1/2} \cos(n\pi x)\,dx + \int_{1/2}^{1} \cos(n\pi x)\,dx$$
$$= 2\int_{1/2}^{1} \cos(n\pi x)\,dx = \frac{2}{\pi n}\sin(n\pi x)\Big|_{1/2}^{1}$$

→ $\cos(n\pi x)$ is <u>even</u>

⇒ $\int_{-1}^{-1/2} \cos(n\pi x)dx = \int_{1/2}^{1} \cos(n\pi x)dx$

↪ $= \frac{-2}{\pi n}\sin\left(\frac{n\pi}{2}\right)$

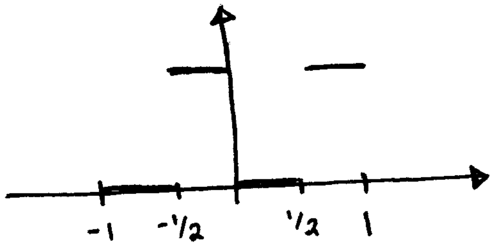
$= 0$ if $n$ is even

$= 1$ if $n = 1 + 4k$ for some $k$

$= -1$ if $n = -1 + 4k$ for some $k$

⇒ Extending by Fourier Cosine Series gives $\sum_{n=1}^{\infty} \underbrace{\frac{-2}{\pi n}\sin\left(\frac{n\pi}{2}\right)\cos(n\pi x)}_{a_n} + \underbrace{1/2}_{a_0 = 1}$

3) Extending by Fourier Series :



Need to extend f in a <u>periodic</u> way

$$\tilde{f}(x) = \begin{cases} 0 & \text{if } x \in (-1, -1/2) \text{ or } (0, 1/2) \\ 1 & \text{if } x \in [-1/2, 0) \text{ or } [1/2, 1) \end{cases}$$

Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(n\pi x) + b_n \sin(n\pi x) \right]$$

$$a_0 = \int_{-1}^{1} \tilde{f}(x)\,dx = 1$$

$$\begin{aligned} a_n &= \int_{-1}^{1} \tilde{f}(x) \cos(n\pi x)\,dx \\ &= \int_{-1/2}^{0} \cos(n\pi x)\,dx + \int_{1/2}^{1} \cos(n\pi x)\,dx \\ &= \frac{\sin(n\pi x)}{n\pi}\bigg|_{-1/2}^{0} + \frac{\sin(n\pi x)}{n\pi}\bigg|_{1/2}^{1} \\ &= \frac{-\sin(-\frac{n\pi}{2})}{n\pi} - \frac{\sin(\frac{n\pi}{2})}{n\pi} = 0 \end{aligned}$$

$$\begin{aligned} b_n &= \int_{-1}^{1} \tilde{f}(x) \sin(n\pi x)\,dx \\ &= \int_{-1/2}^{0} \sin(n\pi x)\,dx + \int_{1/2}^{1} \sin(n\pi x)\,dx \\ &= \frac{1}{n\pi}\left[ -\cos(n\pi x)\bigg|_{-1/2}^{0} - \cos(n\pi x)\bigg|_{1/2}^{1} \right] \\ &= -\frac{1}{n\pi}\left[ 1 - 2\cos\left(\frac{n\pi}{2}\right) + (-1)^n \right] \end{aligned}$$

Extending by Fourier Series gives

$$\underset{\nearrow a_0 = 1}{1/2} + \sum_{n=1}^{\infty} \underbrace{-\frac{1}{n\pi}\left[ 1 - 2\cos\left(\frac{n\pi}{2} + (-1)^n \right]}_{b_n} \sin(n\pi x)$$

MATH

JAN.16/19

- Problem Set 1 posted on D2L
- Expect Assignment 1 (15 % of final mark) on D2L next week, due by end of January.

Recap

- Fourier series of $f:(-p,p)\longrightarrow \mathbb{R}$ is

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{p}\right) + b_n\sin\left(\frac{n\pi x}{p}\right)\right]$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \qquad a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)dx$$

$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{p}\right)$ = Fourier cosine series

$\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{p}\right)$ = Fourier sine series

- convergence of Fourier series

$F(x) = f(x)$ → if $x$ is a continuity point

$F(x) = \frac{f(x-) + f(x+)}{2}$ → if $x$ is a jump point

$f(x\pm) = \lim_{y\to x\pm} f(y)$ → left/right side limits

$F(x)$ extends $f(x)$ periodically

- $f:(0,L)\longrightarrow \mathbb{R}$ → can be extended in 3 ways:
  - Fourier Sine Series
  - Fourier cosine series
  - Fourier series

- Complex Fourier Series : $f(-p,p)\longrightarrow \mathbb{R}$ or $\mathbb{C}$

$$\sum_{n=-\infty}^{+\infty} c_n e^{i\left(\frac{n\pi x}{p}\right)} \qquad c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-i\left(\frac{n\pi x}{p}\right)}dx$$

Today : Sturm-Liouville (SL) problems

- EigenFunction / Eigenvalue
- Sturm - Liouville equation
- Examples

- Eigenvalues / Eigenfunctions

$\boxed{\text{Function}} \xrightarrow[\text{operator } A]{} \boxed{\text{Function}}$

linear operation : $A$ is linear operation if

$A(f + cg) = Af + cAg$

$f, g$ functions, $c \in \mathbb{R}$

$f = g$ means

$f(x) = g(x)$ ⇝ for all $x$

$M$ matrix :

$\underline{\underline{v}} \longmapsto \underline{\underline{Mv}}$

vector

$M(v + cu) = Mv + cMu$

(linear)

Linear operator $\cong$ generation of matrix

$\underbrace{M\underline{u}}_{\text{eigenvectr}} = \underbrace{\underline{\lambda} u}_{\text{eigenvalue}}$ $u \neq 0$ vector

$\underbrace{A\underline{f}}_{\text{eigenfunction}} = \underbrace{\underline{\underline{\lambda}} f}_{\text{eigenvalue}}$ $f \neq 0$ function

$\boxed{\text{Ex}}$ $A = d^2/dx^2$

1) check $A$ is linear operator

2) Find eigenvalues / eigenfunctions

1) Need to check

$$\underset{(f+cg)''}{\underset{\parallel}{A(f+cg)}} = \underset{f''}{\underset{\parallel}{Af}} + \underset{cg''}{\underset{\parallel}{cAg}}$$

⇝ $A$ is just the second derivative

2) Want $Af = \lambda f$ : $f'' = \lambda f$

- if $\lambda > 0$ : $f(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}$
- if $\lambda = 0$ : $f(x) = ax + b$ $a, b \in \mathbb{R}$
- if $\lambda < 0$ : $f(x) = a\cos(\sqrt{|\lambda|}x) + b\sin(\sqrt{|\lambda|}x)$

$\Rightarrow$ All $\lambda \in \mathbb{R}$ are eigenvalues

(with their respective eigenfunction $f(x)$)

## Sturm - Liouville (SL) problem

Sturm-liouville problem

$$\begin{cases} [r(x)y']' + (q(x) + \lambda p(x))y = 0 \\ A_1 y(a) + B_1 y'(a) = 0 \\ A_2 y(b) + B_2 y'(b) = 0 \end{cases} \quad \begin{matrix} \} \text{ Sturm-Liouville equation} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{boundary conditions} \end{matrix}$$

$r(x), p(x) \geq 0$ Functions

$q(x)$ can be positive or negative

$\lambda, A_1, B_1, A_2, B_2, \in \mathbb{R}$

$A_1, B_1,$ **NOT** both $0$ domain $(a, b)$

$A_2, B_2,$ **NOT** both $0$

- can't solve in general

Properties of SL problem:

1) Eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$

   $\lambda_n \to +\infty$ as $n \to +\infty$

2) Each $\lambda_n$ has only **ONE** eigenfunction $y_n$ (upon multiplacative constant)

3) $\{y_n : n = 1, 2, 3, \dots\}$ linearly independent

   if $c_1 y_1 + c_2 y_2 + \dots + c_n y_n + \dots = 0$

   then $c_1 = c_2 = \dots = c_n = 0$

4) $\{y_n : n = 1, 2, 3 \dots\}$ are solutions of SL problems AND its orthogonal set with weight $p(x)$:

   $\int_a^b y_n(x)\, y_m(x)\, p(x) = 0$ whenever $n \neq m$

**Proof of (4):**

$$[r(x) y_n']' + (q(x) + \lambda_n p(x)) y_n = 0$$
$$[r(x) y_m']' + (q(x) + \lambda_m p(x)) y_m = 0$$

Take inner product of 1st equation with $y_m$:

$$\int_a^b [r(x) y_n']' y_m + (q(x) + \lambda_n p(x)) y_n y_m \, dx = 0$$

Take inner product of 2nd equation with $y_n$:

$$\int_a^b [r(x) y_m']' y_n + (q(x) + \lambda_m p(x)) y_m y_n$$

Take difference

$$\int_a^b [r(x)\, y_n']'\, y_m - [r(x)\, y_m']\, y_n \, dx \ \ldots$$

$$\ldots \ + (\lambda_n - \lambda_m) \int_a^b p(x)\, y_n\, y_m \, dx = 0$$

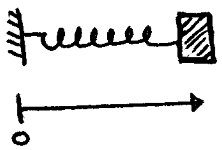
By integration by parts & boundary conditions

$$\int_a^b [r(x)\, y_n']'\, y_m - [r(x)\, y_m']'\, y_n \, dx$$

$$= \int_a^b -r(x)\, y_n'\, y_m' + r(x)\, y_m'\, y_n' \, dx + r(x)\, y_n'\, y_m \Big|_a^b - r(x)\, y_m'\, y_n \Big|_a^b$$

$= 0$ (by boundary conditions)

## Example



$y = y\overset{\text{time}}{t}$ = position of mass

(= distance of mass to wall)

$k$ = spring constant / rest length of spring = 0

$m$ = mass

→ Find equation of motion

Force $= -ky$ (Hooke's Law)

$= ma$ ($a$ = acceleration)

$= m \frac{d^2y}{dt^2}$ (Newton's 2nd law)

$$m \frac{d^2y}{dt^2} = -ky$$

$$\frac{d^2y}{dt^2} = -\frac{k}{m} y \Rightarrow \frac{d^2y}{dt^2} + \frac{k}{m} y = 0$$

this equation of motion is a SL equation with $r(x) = 1$, $q(x) = k/m$, $p(x) = 0$

SL equation

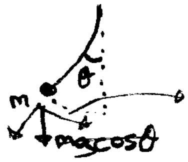
$$[r(x) y']' + (q(x) + \lambda p(x)) y = 0$$

choose : $r(x) = 1$ $q(x) = \frac{k}{m}$ $p(x) = 0$

$$\Rightarrow y'' + \frac{k}{m} y' = 0$$

⊜

Other examples : small amplitude harmonic oscillator (pendulum)



$mg \sin\theta = m\ddot{\theta}$ / $g\theta = \ddot{\theta}$

$\theta \ll 1 \Rightarrow \sin\theta \doteq \theta$ (again, SL eq'n.)