Problem 5: For fixed $t > 0$, find x : $u(x,t) = 0$

$u = f * \underline{\underline{G}}$ heat kernel

$= \int_{\mathbb{R}} f(x-z)\, G(z,t)\, dz$

$= \int_{x-1}^{x+1} \frac{e^{-z^2/4t}}{2\sqrt{\pi t}}\, dz$

$z\,[(x+1)-(x-1)]$ $\qquad \underset{z \in [x-1,\, x+1]}{\min} \frac{e^{-z^2/4t}}{2\sqrt{\pi t}} \ngtr 0$

Recap:

- Intro to Complex Analysis $\quad f : \mathbb{C} \to \mathbb{C}$

  continuity : $f(z) \; \lim_{w \to z} f(w)$

  differentiability : $f(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$

  analytic : $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$

- e^z, polynomials of z, $\mathbb{C} \to \mathbb{C}$ ("entire function")

- $\frac{p(z)}{q(z)}$ p, q polynomials, are differentiable whenever $q(z) \neq 0$ ("meromorphic")

- differentiable $\{z : |z - z_0| \leq r\} \longrightarrow \mathbb{C}$ ("holomorphic")

entire $\to$ meromorphic $\longrightarrow$ holomorphic

$\quad\;\; \not\leftarrow \qquad\qquad\qquad \not\leftarrow$

Picard theorem : $f : \mathbb{C} \to \mathbb{C}$ entire

then $f(\mathbb{C}) \begin{cases} \mathbb{C} & (f \text{ subjective}) \\ z & (f \text{ constant}) \\ \mathbb{C} \setminus \{z_0\} & \end{cases}$

- Complex Integral : $f : \mathbb{C} \to \mathbb{C}$

  path $\gamma : [0, T] \to \mathbb{C}$

$\int_\gamma f(z)\, dz = \int_0^T f(\gamma(t))\, |\gamma'(t)|\, dt$

Today: 1) Cauchy - Goursat Theorem
2) Cauchy integral formula

- Cauchy - Goursat theorem

"holomorphic functions are conservative"

$f : B(z_0, r) \to \mathbb{C}$

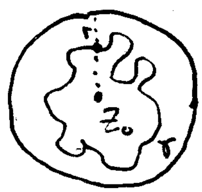
$\{ z : |z - z_0| < r \}$

"open ball"

Path

$\gamma : [0, T] \to B(z_0, r)$

then $\int_\gamma f(z)\, dz = 0$

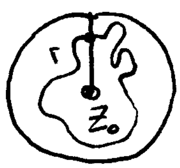


$\xrightarrow{f \text{ holomorphic},} \mathbb{C}$

then

$\int_\gamma f(z) dz = 0$

- Cauchy integral Formula



$\xrightarrow{f} \mathbb{C}$

f holomorphic on all the ball $B(z_0, r)$ <u>except</u> at most z_0

- $\gamma : [0, T] \to \underbrace{B(z_0, r) \setminus \{z_0\}}$

avoids the center z_0

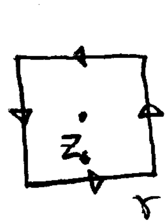
Then,

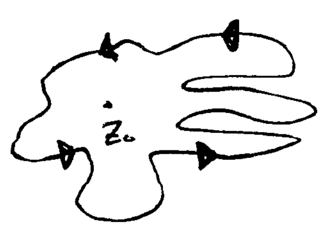
$$\int_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0) \cdot \underbrace{\text{"winding number of } \gamma\text{"}}_{\text{how many loops } \gamma \text{ makes around } z_0}$$

$$\int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0) \quad \text{winding \# of } \gamma$$

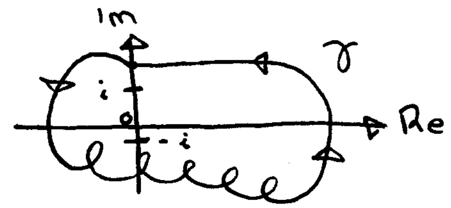
$(n = 1, 2, 3, \dots)$

1) $\int_\gamma \frac{f(z)}{z - z_0}\, dz$ depends <u>only</u> on $f(z)$ and winding # of γ





Ex: Find $\int_\gamma \frac{z}{z^2+1}\, dz$



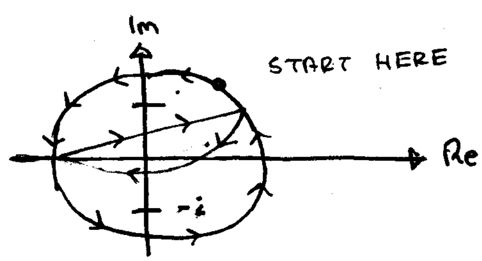
- γ loops <u>once</u> around $\pm i$

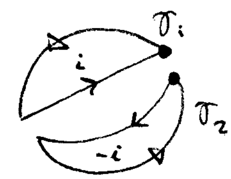
$$\int_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0) \cdot \text{winding number of } \gamma$$

$\hookrightarrow$ z_0 is point where denominator $= 0$

But denominator of $\frac{z}{z^2+1}$ (i.e. z^2+1)

is 0 at both $\pm i$ ...





$$\rightarrow \int_\gamma \frac{z}{z^2+1}\, dz \underset{\gamma_1 + \gamma_2}{=} \int_{\gamma_1} \frac{z}{z^2+1}\, dz + \int_{\gamma_2} \frac{z}{z^2+1}\, dz$$

$\pm i$ once! (just like γ)

- γ_1 loops around i once:

$$\int_{\gamma_1} \frac{f_1(z)}{z - i}\, dz = 2\pi i f_1(i)$$

$= i = z_0$ from Cauchy Integral

$$\frac{z}{z^2+1} = \frac{f_1(z)}{z - i}$$

$$\downarrow$$

$$\frac{z}{(z+i)(z-i)} \longrightarrow f_1(z) = \frac{z}{2z+i}$$

$$\int_{\gamma_2} \frac{z}{z^2+1}\, dz = \int_{\gamma_2} \frac{z}{(z+i)(z-i)}\, dz \qquad f_i(z) = \frac{z}{z-i}$$

$$= \int_{\gamma_2} \frac{z/z-i}{z+i}\, dz \longrightarrow -i = z_0 \text{ from Cauchy integral}$$

$$= 2\pi i\, f(-i) = \pi i$$

$\rightarrow$ original $\int_\gamma \frac{z}{z^2+1}\, dz = 2\pi i$

Recap:

- Cauchy - Goursat theorem

$f : B(z_0, r) \to \mathbb{C}$ holomorphic

$\gamma : [0, T] \to B(z_0, r) \to \int_\gamma f(z)\, dz = 0$

- Cauchy integral formula

$f : B(z_0, r) \to \mathbb{C}$ holomorphic except at most at z_0

$\gamma : [0, T] \to B(z_0, r)$

AVOIDING z_0

$\to \int_\gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0)$ winding # of γ

$\hookrightarrow$ z_0 is "POLE"

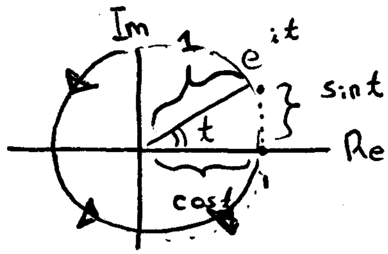
Review :

5. Find $\displaystyle\int_c \frac{e^{z^2}}{z-2}\,dz$

$C = \gamma([0, 2\pi])$

$\gamma(t) = \underbrace{6i + 4}_{\text{does not depend on } t} + \underbrace{e^{it}}_{\text{depends on } t}$

→ Acts like a translation



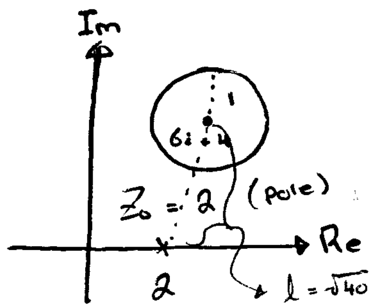
$e^{it} = \cos t + i \sin t$

$t \in [0, 2\pi]$

e^{it}, $t \in [0, 2\pi]$ is a circle with center 0, radius 1

$+ 6i + 4$

center $6i + 4$, radius 1



Pole $Z_0 = 2$ is outside C (greater than radius)

Distance between 2 and $6i + 4$ is

$|2 - 6i - 4| = |-6i - 2|$

$= \sqrt{2^2 + 6^2} = \sqrt{40}$

$\sqrt{40} > 1$ (radius of C)

→ $\dfrac{e^{z^2}}{z-2}$ has NO poles in C

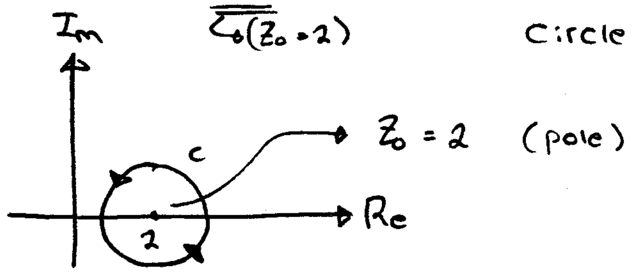
→ by Cauchy - Goursat theorem

$$\int_c \frac{e^{z^2}}{z-2} = 0$$

4 Find $\int_C \frac{e^{z^2}}{\underset{\zeta_0(z_0 = 2)}{z-2}}\,dz$ $\qquad C = \gamma([0, 2\pi])$ ← if this was 4π, that would be 2 loops (winding # = 2)

$\gamma(t) = 2 + e^{it}$

Circle with center 2, radius 1



because the pole is inside:

$$\int_C \frac{e^{z^2}}{z-2}\,dz = \int_C \frac{f(z)}{z - z_0}\,dz$$

$$= 2\pi i \underbrace{f(z_0)}_{e^4} \cdot \underbrace{\text{winding \# of } C}_{=1} \qquad \begin{cases} f(z) = e^{z^2} \\ z_0 = 2 \end{cases}$$

$$= 2\pi i e^4$$

3 Find Fourier Transform of

$$f(t) = \begin{cases} e^t & \text{if } t \in [0, 1] \\ 0 & \text{if not} \end{cases}$$

$$\mathcal{F}[f_1](\omega) = \int_{\mathbb{R}} f(t)\, e^{-i\omega t}\,dt = \int_0^1 e^t e^{-i\omega t}\,dt$$

$$= \int_0^1 e^{t(1-i\omega)}\,dt = \left.\frac{e^{t(1-i\omega)}}{1-i\omega}\right|_0^1 \quad \text{← don't substitute for } \omega$$

$$= \frac{e^{1-i\omega} - 1}{1 - i\omega}$$

2 Find $\int_{-\infty}^{+\infty} e^{-x^2/5}\,dx$

$$\int_{-\infty}^{+\infty} e^{-x^2/5}\,dx = \sqrt{\int_{-\infty}^{+\infty} e^{-x^2/5}\,dx}\ \sqrt{\int_{-\infty}^{+\infty} e^{-y^2/5}\,dy}$$

$$= \sqrt{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2/5} e^{-y^2/5}\,dy\,dx}$$

$$= \sqrt{\int_0^{2\pi}\int_0^{+\infty} \underbrace{e^{-\rho^2/5}\rho\,d\rho}_{\left.-\frac{5}{2}e^{-\rho^2/5}\right|_0^{+\infty} = \frac{5}{2}}\,d\theta} = \sqrt{5\pi}$$

integration of θ gives 2π

1 $u_{tt} = 4u_{xx}$ Subject to:

$\lim_{x\to\pm\infty} u(x,t) = 0$

$u(x,0) = e^{-x^2}\sin 2x$

$u_t(x,0) = e^{-x^4}$

Solve using Fourier

i) Take $\mathcal{F}$ (in x) $\quad (\hat{u} = \mathcal{F}[u])$

$\hat{u}_{tt} = 4(i\omega)^2\hat{u} = -4\omega^2\hat{u}$

$\rightarrow \hat{u}(\omega,t) = \underbrace{A(\omega)}_{\text{to find}}e^{i2\omega t} + \underbrace{B(\omega)}_{\text{to find}}e^{-i2\omega t}$

$\hat{u}(\omega,0) = \int_{\mathbb{R}} u(x,0)e^{-i\omega x}\,dx = \widehat{u(x,0)}(\omega)$

$= \widehat{\underbrace{e^{-x^2}\sin(2x)}_{f(x)}}$ (under the hat)

$\hat{u}_t(\omega,0) = \int_{\mathbb{R}} u_t(x,0)e^{-i\omega x}\,dx$

$= \widehat{u_t(x,0)} = \widehat{\underbrace{e^{-x^4}}_{g(x)}}$ (under the hat)

$\hat{u}(\omega,0) = A(\omega) + B(\omega) = \hat{f}(\omega)$

$\hat{u}_t(\omega,0) = 2\omega i\,[A(\omega) - B(\omega)] = \hat{g}(\omega)$

$$A(\omega) = \frac{\hat{g}(\omega)}{2\omega i} + B(\omega)$$

$$\rightarrow \frac{\hat{g}(\omega)}{2\omega i} + 2B(\omega) = \hat{f}(\omega)$$

$$B(\omega) = \frac{\hat{f}(\omega)}{2} - \frac{\hat{g}(\omega)}{4\omega i}$$

$$A(\omega) = \frac{\hat{f}(\omega)}{2} + \frac{\hat{g}(\omega)}{4\omega i}$$

$$\rightarrow \hat{u}(\omega,t) = \underbrace{\left[\frac{\hat{f}(\omega)}{2} + \frac{\hat{g}(\omega)}{4\omega i}\right]}_{A(\omega)}e^{2\omega i t} + \underbrace{\left[\frac{\hat{f}(\omega)}{2} - \frac{\hat{g}(\omega)}{4\omega i}\right]}_{B(\omega)}e^{-2\omega i t}$$

ii) Take $\mathcal{F}^{-1}$

$\mathcal{F}^{-1}[\hat{u}] = u(x,t)$

$$= \underbrace{\mathcal{F}^{-1}\left[\frac{\hat{f}(\omega)}{2}e^{2\omega i t}\right]}_{\text{I}} + \underbrace{\mathcal{F}^{-1}\left[\frac{\hat{f}(\omega)}{2}e^{-2\omega i t}\right]}_{\text{II}} + \underbrace{\mathcal{F}^{-1}\left[\frac{\hat{g}(\omega)}{2}e^{2\omega i t}\right]}_{\text{III}} + \underbrace{\mathcal{F}^{-1}\left[\frac{\hat{g}(\omega)}{2}e^{-2\omega i t}\right]}_{\text{IV}} -$$

I $$F^{-1}\left[\frac{\hat{f}(\omega)}{2}\, e^{2\omega i t}\right] = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\hat{f}(\omega)}{2}\, e^{2\omega i t}\, e^{i x\omega}\, d\omega$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\hat{f}(\omega)}{2}\, e^{i\omega(x+2t)}\, d\omega = \frac{f(x+2t)}{2}$$

II $$F^{-1}\left[\frac{\hat{f}(\omega)}{2}\, e^{-2\omega i t}\right]$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\hat{f}(\omega)}{2}\, e^{i\omega(x-2t)}\, d\omega = \frac{f(x-2t)}{2}$$

III $$F^{-1}\left[\frac{\hat{g}(\omega)}{4\omega i}\, e^{2\omega i t}\right]$$

$$= \frac{1}{4}\cdot\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\hat{f}(\omega)}{2}\, e^{i\omega(x+2t)}\, d\omega$$

$$= \left(\frac{1}{4}\right) F^{-1}\left[\frac{\hat{g}(\omega)}{i\omega}\right](x+2t) = \left(\frac{1}{4}\right) G(x+2t)$$

$$\| \; G(y) = \int_{-\infty}^{y} e^{(z)}\, dz$$
antiderivative

IV $$F^{-1}\left[\frac{\hat{g}(\omega)}{4\omega i}\, e^{-2\omega i t}\right]$$

$$= \frac{1}{4}\cdot\frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\omega)\, e^{i\omega(x-2t)}\, d\omega = \left(\frac{1}{4}\right) G(x-2t)$$

$$\rightarrow u(x,t) = \left(\tfrac{1}{2}\right)\left[f(x+2t) + f(x-2t) + \left(\tfrac{1}{2}\right)G(x+2t) - \left(\tfrac{1}{2}\right)G(x-2t)\right]$$

where $f(x) = e^{-x^2}\sin 2x$

$$= \left(\tfrac{1}{2}\right)\Big[e^{-(x+2t)^2}\sin(2x+4t) + e^{-(x-2t)^2}\sin(2x-4t) \;\cdots$$

$$\cdots\; \underbrace{+ \left(\tfrac{1}{2}\right)\int_{-\infty}^{x+2t} e^{-z^4}\, dz - \left(\tfrac{1}{2}\right)\int_{-\infty}^{x-2t} e^{-z^4}\, dz}_{= \left(\frac{1}{2}\right)\int_{x-2t}^{x+2t} e^{-z^4}\, dz}$$