Recap:

- Numerical methods for PDEs

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \qquad h \ll 1$$

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

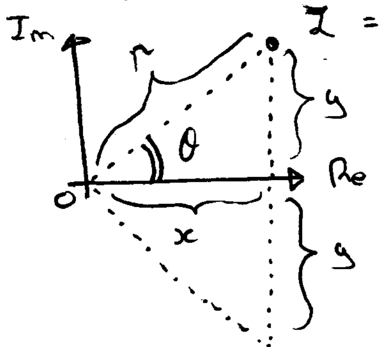
- Laplace eq'n: $u_{xx} + u_{yy} = 0$:
  $u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h) = 4u(x, y)$

- Heat eq'n $u_t = k u_{xx}$
  $$u(x, t+h) = -u(x,t) + k\,\frac{u(x+h, t) - 2u(x,t) + u(x-h,t)}{h}$$

- Wave eq'n $u_{tt} = c^2 u_{xx}$
  $u(x, t+h) - 2u(x,t) + u(x, t-h)$
  $= c^2 [u(x+h, t) - 2u(x,t) + u(x-h), t)]$

Today: Intro to complex Analysis

Complex Analysis studies functions with complex variables



$z = x + iy = re^{i\theta}$

$r = \sqrt{x^2+y^2}$

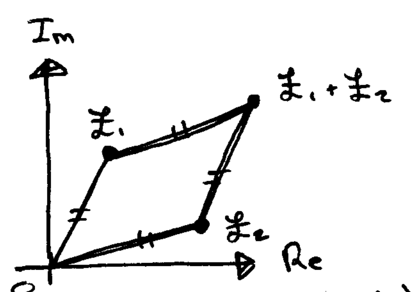
$\theta = \arctan(y/x)$

$x = r\cos\theta$

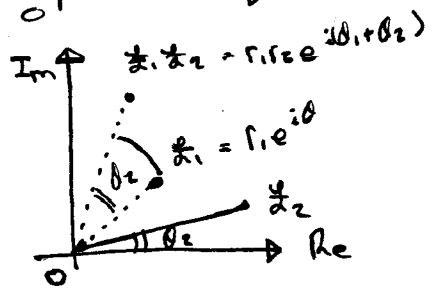
$y = r\sin\theta$

$\bar{z} = x - iy = re^{-i\theta}$

$\bar{z}$ = conjugate of z



Sum ~ done by "parallelogram rule"



Product of norms

Sum of angles

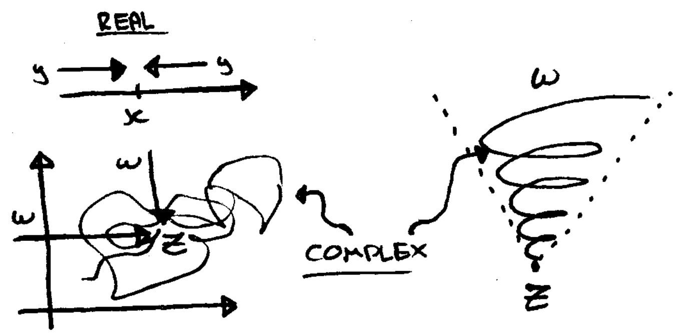
Complex Functions $f: \mathbb{C} \to \mathbb{C}$

- f <u>continuous</u> at $z \in \mathbb{C}$ if
  $f(x) = \lim_{\omega \to z} f(\omega)$ (just like for functions)

- Differentiability:
  real: $f'(x) = \lim_{x \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{y \to x} \dfrac{f(y) - f(x)}{y - x}$

complex: $f'(x) = \underbrace{\lim_{\omega \to z} \dfrac{f(\omega) - f(z)}{\omega - z}}$

→ definition identical to real case



- complex differentiability
  $\neq$ real $\underbrace{\text{analiticity}}$

$g: \mathbb{R} \to \mathbb{R}$ is analitic if
$$g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(x_0)}{n!} (x - x_0)^n$$

- If $f: \mathbb{C} \to \mathbb{C}$ is differentiable at all $z \in \mathbb{C}$ then
$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

- f differentiable on some ball
  $\{ |\omega - z| \leq r \}$ → f HOLOMORPHIC
- f differentiable for all $z \in \mathbb{C}$ <u>except</u>
  for isolated values → f MEROMORPHIC
- f differentiable for all $z \in \mathbb{C}$ → f ENTIRE

e^z, polynomials of z are ENTIRE

$\dfrac{p(z)}{q(z)}$ (p, q polynomials) is MEROMORPHIC

differentiable everywhere
<u>except</u> where $q(z) = \emptyset$

$f(r,\theta) = \theta$ only Holomorphic

Sum of holomorphic functions is holomorphic

- Picard's theorem : $f: \mathbb{C} \to \mathbb{C}$ entire
  then image $f(\mathbb{C})$ can be $\begin{cases} \mathbb{C} \\ \{z_0\} \\ \mathbb{C} \setminus \{z_0\} \end{cases}$

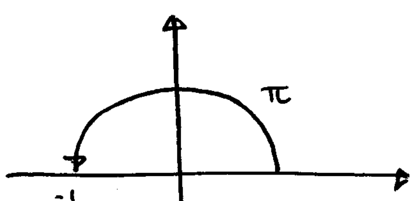
<u>Very</u> different from real functions

- real analytic functions can have image $[0, +\infty)$ e.g. $f(x) = x^2$
- real analytic functions <u>cannot</u> have image $\mathbb{R} \setminus \{x_0\}$ for some $x_0 \in \mathbb{R}$

<u>Ex</u> : $f(z) = e^z$ take all values except 0

now to choose z : $f(z) = re^{i\theta}$

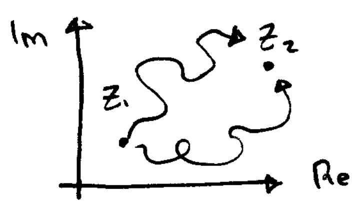
$\left.\begin{matrix} f(z) = e^z = re^{i\theta} \\ e^a e^{ib} \; (z = a+ib) \end{matrix}\right\}$ take $\begin{matrix} a = \ln r & (r \neq 0) \\ b = \theta & \end{matrix}$



$e^{i\pi} = -1$

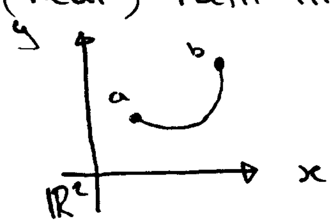
z^{2m} not ensured to be non-negative anymore...

- Integral : $\int_a^b f(x)dx$ (Real case)
  complex : <u>NOT</u> $\int_{z_1}^{z_2} f(z)dz$ (makes no sense)



PATH MUST BE SPECIFIED!

(Real) Path integral :



$\sigma : [0, T] \to \mathbb{R}^2$

path between $a, b \in \mathbb{R}^2$

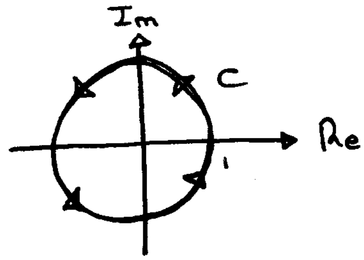
$\int_\sigma f(x)dx =$

$\int_0^T f(\sigma(t))\,|\sigma'(t)|\,dt$

Complex Integral : $\sigma : [0, T] \to \mathbb{C}$

$$\int_\sigma f(z)\,dz = \int_0^T f(\sigma(t))\,|\sigma'(t)|\,dt$$

Ex :



Find $\int_C z^3\,dz$

we need first a parameterization σ

$\sigma : [0, 2\pi) \to \mathbb{C}$

$\sigma(t) = e^{it}$

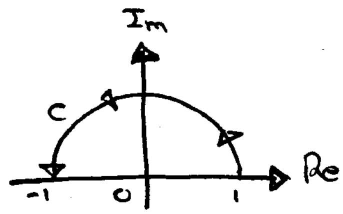
$$\int_C z^3\,dz = \int_0^{2\pi} \sigma(t)^3 \underbrace{|\sigma'(t)|\,dt}_{\sigma(t) = e^{it}}$$

$$|\sigma'(t)| = |ie^{it}| = |i| \cdot |e^{it}| = 1$$

$$\to \; = \int_0^{2\pi} e^{3it}\,dt$$

$$= \frac{e^{3it}}{3i}\Bigg|_0^{2\pi} = \frac{e^{6\pi i} - 1}{3i} = 0$$

Ex :



Find $\int_C e^z\,dz$

$\sigma : [0, \pi] \to \mathbb{C}$

$\sigma(t) = e^{it}$

$$\int_0^\pi e^z\,dz = \int_0^\pi e^{e^{it}} \underbrace{|\sigma'(t)|\,dt}_{=1}$$

use $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $(n! = n(n-1)\cdots 3 \cdot 2 \cdot 1)$

$$\to \int_0^\pi e^{e^{it}}\,dt = \int_0^\pi \sum_{n=0}^{\infty} \frac{e^{nit}}{n!}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^\pi e^{nit}\,dt$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) \frac{e^{nit}}{ni}\Bigg|_0^\pi = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) \frac{\overbrace{e^{in\pi}}^{(-1)^n} - 1}{ni}$$

$$= \sum_{n \text{ odd}} \frac{-2}{n!\,ni}$$