$x_1^2 + x_1 x_2 = 10$
$x_2 + 3x_1 x_2^2 = 57$

MATH

(1)

MAR. 11 /19

Recap :

- $\mathcal{F}$ to solve PDEs
  1) Take $\mathcal{F}$ (1/3)
  2) Computations (1/3)
  3) Take $\mathcal{F}^{-1}$ (1/3)

- Heat eq'n $u_t = k u_{xx}$ $(k > 0)$

$\lim_{x \to \pm\infty} u(x,t) = 0, \quad u(x,0) = f(x)$

(1) $\mathcal{F}[u_t] = \hat{u}_t = k\mathcal{F}[u_{xx}] = -k\omega^2 \hat{u}$

(2) ODE in $\hat{u}$: solution $\hat{u}(\omega, t) = A(\omega) e^{-k\omega^2 t}$

$\hat{u}(\omega, 0) = \mathcal{F}[u(x,0)] = \hat{f}(\omega)$

$\longrightarrow \hat{u} = \hat{f}(\omega) e^{-k\omega^2 t}$

(3) $\mathcal{F}(f * g) = \mathcal{F}[f]\,\mathcal{F}[g]$

$(f * g)(x) = \int_{\mathbb{R}} f(z) g(x-z) dz$

$\hat{u} = \hat{f} \cdot \mathcal{F}[\mathcal{F}^{-1}[e^{-k\omega^2 t}]] \longrightarrow u = f * G$

$= G$ "heat kernal" (solution)

- Wave eq'n $u_{tt} = c^2 u_{xx}$ $(c > 0)$

$\lim_{x \to \pm\infty} u(x,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)$

(1) Take $\mathcal{F}$: $\underbrace{\hat{u}_{tt}}_{\mathcal{F}[u_{tt}]} = \underbrace{-c^2\omega^2 \hat{u}}_{\mathcal{F}[c^2 u_{xx}]}$

(2) ODE with solution

$\hat{u} = A(\omega) \cos c\omega t + B(\omega) \sin c\omega t$

$= C(\omega) e^{ic\omega t} + D(\omega) e^{-ic\omega t}$

$$\left.\begin{aligned} C(\omega) + D(\omega) &= \hat{f}(\omega) \\ ic\omega [C(\omega) - D(\omega)] &= \hat{g}(\omega) \end{aligned}\right\} \text{ can find } C(\omega), D(\omega)$$

(3) $u = \frac{1}{2\pi} \int_{\mathbb{R}} \left[\frac{1}{2}\hat{f}(\omega) - \frac{i}{2c\omega}\hat{g}(\omega) \right] e^{i\omega(x+ct)} d\omega \ \cdots$

$\cdots + \frac{1}{2\pi} \int_{\mathbb{R}} \left[\frac{1}{2}\hat{f}(\omega) - \frac{i}{2c\omega}\hat{g}(\omega) e^{i\omega(x-ct)} d\omega \right.$

$\boxed{\text{TODAY}}$: Complete wave equation
Discrete Fourier Transform

End of Step 2 :

$C(\omega) = \left(\frac{1}{2}\right)\hat{f}(\omega) - \frac{i}{2c\omega}\hat{g}(\omega)$

$D(\omega) = \left(\frac{1}{2}\right)\hat{f}(\omega) + \frac{i}{2c\omega}\hat{g}(\omega)$

$\longrightarrow \hat{u} = C(\omega) e^{ic\omega t} + D(\omega) e^{-ic\omega t}$

(3) Take $\mathcal{F}^{-1}$:

$$u = \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}} \underbrace{\frac{1}{2}\hat{f} - \frac{i}{2c\omega}\hat{g}}_{C(\omega)} \Big) e^{ic\omega t} e^{i\omega x} d\omega + \left(\frac{1}{2\pi}\right) \int_{\mathbb{R}} \underbrace{\left(\frac{1}{2}\hat{f} + \frac{i}{2c\omega}\hat{g}\right)}_{D(\omega)} e^{-ic\omega t} e^{i\omega x} d\omega$$

$\cdot\ \left(\frac{1}{2}\right)\underbrace{\left\{\frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\omega)\, e^{i\omega(x\pm ct)}\, d\omega\right\}}_{=\ F^{-1}[\hat{f}](x\pm ct)} = \left(\frac{1}{2}\right) f(x\pm ct)$

$\cdot\ \left(\frac{1}{2c}\right)\left\{\frac{1}{2\pi}\int_{\mathbb{R}} \frac{-i}{\omega}\,\hat{g}(\omega)\, e^{i\omega(x+ct)}\, d\omega\right\}$

$F[U_x] = i\omega\hat{u}$ & $F[u^{(n)}] = (i\omega)^n\hat{u}$

$F[U] = \frac{1}{i\omega}\hat{u}$ & $U(x) = \int_{-\infty}^{x} u(z)\, dz$

anti-derivative

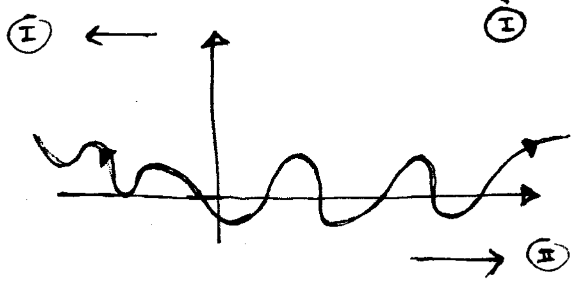
$= \frac{1}{2c}F^{-1}\left[\frac{-i}{\omega}\hat{g}\right](x+ct) = \frac{1}{2c}G(x+ct)$

$\left(G(x) = \int_{-\infty}^{x} g(x)\, dz\right)$

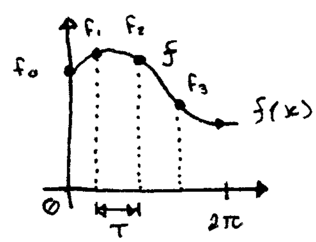
$\cdot\ \frac{1}{2c}\left\{\frac{1}{2\pi}\int_{\mathbb{R}} \frac{i}{\omega}\,\hat{g}(\omega)\, e^{i\omega(x-ct)}\, d\omega\right\} = \frac{-1}{2c}G(x-ct)$

Solution:

$$u(x,t) = \underbrace{\frac{1}{2}\left(f(x+ct) + \frac{1}{c}G(x+ct)\right)}_{\mathrm{I}} + \underbrace{\frac{1}{2}\left(f(x-ct) - \frac{1}{c}G(x-ct)\right)}_{\mathrm{II}}$$



Discrete Fourier Transform (DFT):



$f: (0, 2\pi) \to \mathbb{R}$

its Fourier series is $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ \cdots$

$\cdots + \sum_{n=1}^{\infty} b_n \sin nx$

$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\, dx$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\, dx$

Complex Fourier series: $\cos nx + i\sin nx = e^{inx}$

$F(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$

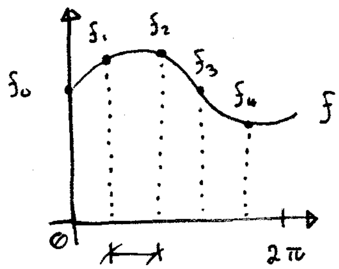
$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{inx}\, dx$

$\sum_{n=0}^{N} f(x)\,\delta(x-nT)$ sampled function

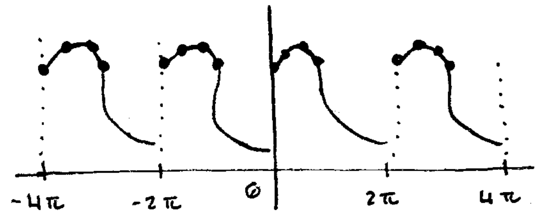
T = "sampling rate" $N = \left[\frac{2\pi}{T}\right]$

it's Fourier series is

$\underbrace{F(\alpha) = \sum_{n=-\infty}^{+\infty} f(nT)\, e^{i\alpha nT}}$ "Discrete Fourier Transform" (of f)



$\sum f(x)\,\delta(x-nT)$



$F(x)$ "sampled function extended periodically"

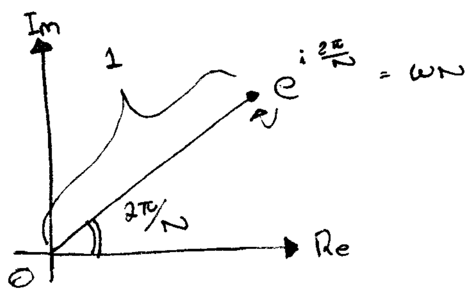
- Sampling is "lossy"
  (can't reconstruct f from sampled function)
- But sampled functions can be treated / analyzed more efficiently

$$\boxed{\begin{matrix}\text{Function}\\ f\end{matrix}} \xrightarrow{\text{sampling}} \boxed{\vec{f} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix} \quad \text{vector}}$$

AND :

$$\underbrace{\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ f_n \end{bmatrix}}_{\vec{f}} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \cdots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \cdots & \omega_N^{(N-1)^2} \end{bmatrix}}_{\boxed{F_N}} \underbrace{\begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}}_{\vec{c}\ \to \left(\begin{matrix}\text{Fourier}\\ \text{coeff.}\end{matrix}\right)}$$

$$\omega_N = e^{i\frac{2\pi}{N}}$$



$F_N \overline{F_N} = N\,id$

depends only on N

(NOT on $\vec{f}, \vec{c}$)

$\vec{f} = F_N \vec{c}$ $\qquad$ $\vec{c} = F_N^{-1} \vec{f}$

DFT pair $\qquad$ $F_N^{-1} = \frac{1}{N} \overline{F_N}$

- Very efficient to pass $\vec{c} \longleftrightarrow \vec{f}$
- Baud limited samples :

  frequency $\underbrace{\omega \in (-A, A)}_{\text{bounded frequency}}$

$$\text{Then : } f(x) = \underbrace{\sum_{n=-\infty}^{+\infty}}_{\substack{\text{finite sum}\\ \text{(not series)}}} \underbrace{f\left(\frac{n\pi}{A}\right)}_{\text{samples}} \frac{\sin(Ax - n\pi)}{Ax - n\pi}$$

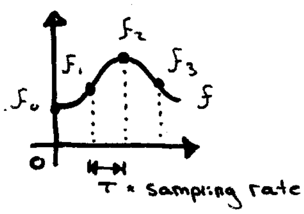
$$\underbrace{\frac{n\pi}{A}}_{\text{sampling rate}} = n \cdot \overbrace{\boxed{\frac{2\pi}{A}}}^{\text{highest frequency}} \cdot \frac{1}{2}$$

→ need only to sample 2x/as the highest freq. / per period

March 13/17

- Assignment 3 posted on D2L – due 11.59pm, March 28TH

RECAP :



$f(0, 2\pi) \longrightarrow \mathbb{R}$

$\sum f(nT)\,\delta(x - nT)$ Sampled Function

$F(x) = \sum_{n=-\infty}^{+\infty} C_n e^{inaT}$ (DFT)

$C_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(x)\, e^{-inax}\, dx$

- Advantage : $\boxed{\text{Function } f} \xrightarrow[\text{sampling}]{} \boxed{[f_0, f_1, \ldots, f_n}$

$$\underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}}_{\vec{f}} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \cdots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \cdots & \omega_N^{(N-1)^2} \end{bmatrix}}_{F_N} \underbrace{\begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}}_{\vec{c}}$$

$\omega_N = e^{i2\pi/N}$

$\vec{f} = F_N \vec{c}$ $\qquad \vec{c} = F_N^{-1}\vec{f}$ $\qquad \left(F_N^{-1} = \frac{1}{N}\overline{F_N}\right)$

DFT Pair $\qquad F_N$ is $\underline{N \times N}$ matrix

i^{th} row
j^{th} column $\quad : \quad \omega_N^{(i-1)(j-1)}$

- Band limited Function f :
  freq. $\in (-A, A)$ then

$$f(x) = \sum_{n=-\infty}^{+\infty} f\left(\frac{n\pi}{A}\right) \frac{\sin(Ax - n\pi)}{Ax - n\pi}$$

Nyquist-Shannon Sampling Theorem "Sample 2x/period at the highest Freq."

Today: 1) sketch of Proof
2) Intro to numerical methods

Function f : freq. $\in (-A, A)$

$F(\alpha) = \int_{\mathbb{R}} f(x)\, e^{i\alpha x}\, dx$

$f(x) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} F(\alpha)\, e^{-i\alpha x} = \dfrac{1}{2\pi}\underbrace{\int_{-A}^{A}}_{\text{band limited}} F(\alpha)\, e^{-i\alpha x}\, d\alpha$

$F(\alpha) = \sum_{n=-\infty}^{+\infty} C_n e^{in\left(\frac{\alpha}{A}\right)\pi}$

$$C_n = \frac{\pi}{A} \cdot \underbrace{\frac{1}{2\pi} \int_{\mathbb{R}} F(\alpha) e^{-in\pi\left(\frac{\alpha}{A}\right)} d\alpha}_{= f\left(\frac{n\pi}{A}\right)}$$

Thus,

$$f(x) = \frac{1}{2\pi} \int_{-A}^{A} F(\alpha) e^{-i\alpha x} d\alpha$$

$$= \frac{1}{2\pi} \int_{-A}^{A} \underbrace{\left[\sum_{n=-\infty}^{+\infty} C_n e^{\frac{in\pi\alpha}{A}} \right]}_{= F(\alpha)} e^{-i\alpha x} d\alpha$$

$$= \frac{1}{2\pi} \int_{-A}^{A} \sum_{n=-\infty}^{+\infty} \frac{\pi}{A} f\left(\frac{n\pi}{A}\right) e^{in\pi\left(\frac{\alpha}{A}\right)} e^{-i\alpha x} d\alpha$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \frac{\pi}{A} f\left(\frac{n\pi}{A}\right) \int_{-A}^{A} e^{i\alpha\left(\frac{n\pi}{A} - x\right)} d\alpha$$

$$= \frac{1}{2A} \sum_{n=-\infty}^{+\infty} f\left(\frac{n\pi}{A}\right) \left[\frac{e^{i\alpha\left(\frac{n\pi}{A} - x\right)}}{i\left(\frac{n\pi}{A} - x\right)} \right]_{-A}^{A}$$

$$= \frac{1}{A} \sum_{n=-\infty}^{+\infty} f\left(\frac{n\pi}{A}\right) \left[\underbrace{\frac{e^{iA\left(\frac{n\pi}{A} - x\right)} - e^{-iA\left(\frac{n\pi}{A} - x\right)}}{2i} \cdot \frac{1}{\frac{n\pi}{A} - x}} \right]$$

$$\frac{e^{iz} - e^{-iz}}{2i} = \sin z, \quad z = n\pi - Ax$$

$$= \sum_{n=-\infty}^{+\infty} f\left(\frac{n\pi}{A}\right) \frac{\sin(n\pi - Ax)}{n\pi - Ax} = f(x)$$

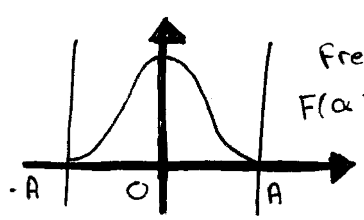
→ Nyquist - Shannon Theorem has "bad cases"

Ex. $f(x) = \sin Ax$

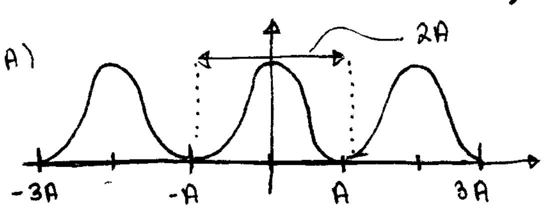
$$\rightarrow \sum_{n=-\infty}^{+\infty} \underbrace{f\left(\frac{n\pi}{A}\right)}_{\sin\left(A \cdot \frac{n\pi}{A}\right) = 0} \frac{\sin(n\pi - Ax)}{n\pi - Ax} = 0 \neq f(x)$$

aliasing effects

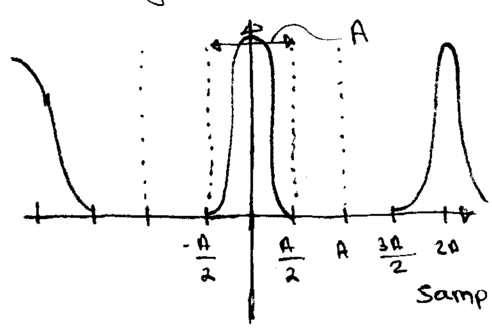
- Sampling rate is often pre-fixed (e.g. industrial standards)



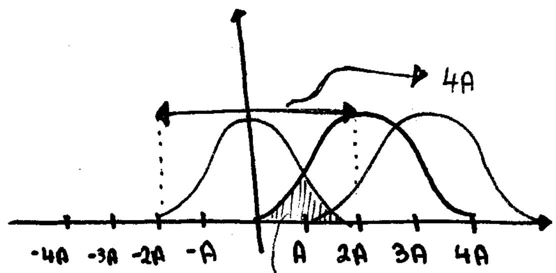
band limited signal



Sampling at 2x / period



Sampling at 4x / period



Aliasing

(Frequency bands overlapping)

Sampling at 1x / period

- Low bypass Filters

$F(\alpha)$ freq. of signal f

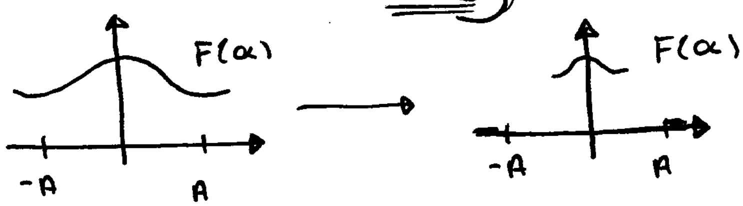
$F(\alpha)G(\alpha)$ is band limited $\in (-A, A)$

$$G(\alpha) = \begin{cases} 1 & \text{if } \alpha \in (-A, A) \\ 0 & \text{if not} \end{cases}$$

$G = \mathcal{F}[g] \; (g = \mathcal{F}^{-1}[G])$

$\rightarrow F(\alpha)G(\alpha)$ is freq. of signal

$\mathcal{F}^{-1}[FG] = \underline{f * g}$ low freq. filter



high frequency part "removed"

$\vec{f} = \underbrace{F_N}_{} \vec{c}$

$\hookrightarrow$ $N \times N$ matrix

$\sim O(N^2)$ computations ...

Fast-Fourier transform

when $N = 2^m$

then exist algorithms

computing F_N in

$O(N \ln N)$ computations

$\underline{N \ln N \ll N^2}$

Introduction to Numerical Methods

$$f'(x) = \lim_{h \to 0} \boxed{\frac{f(x+h) - f(x)}{h}} \quad h \ll 1$$

Finite difference approximation

$$f''(x) = \lim_{h \to 0} \boxed{\frac{f(x+h) - 2f(x) + f(x-h)}{h^2}} \quad h \ll 1$$