Jan. 8 /18

Grade comp:

- 3x take home assignment. (15 %) each, (45 % total)
- & Final exam (55 %)

- No make-up assignments
- No extensions (except for documented reasons)

→ Practice problems will be provided (but will not be graded)

Monday / wednesday : 1 → 2pm (office hours)
↳ RB 2015

Important topics :

- Differentiation :
  - Basic derivatives of
    x^n, $\sin x$, $\cos x$, e^x, $\ln x$
  - Rules of derivation :

    linearity : $\frac{d}{dx}\left(f(x) + cg(x)\right)$
    $= f'(x) + cg'(x)$, c = constant

    Product : $(f(x) \cdot g(x))'$
    $= f(x)g'(x) + f'(x)g(x)$

    chain rule : $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

    quotient rule : $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - g'(x)f'(x)}{g(x)^2}$

- Integration :
  - Basic integrals of
    x^n, $\sin x$, $\cos x$, e^x, $\ln x$
  - Rules of integration :

    linearity : $\int [f(x) + cg(x)]\,dx$
    $= \int f(x)\,dx + c\int g(x)\,dx$

Substitution

$y = g(x)$, then $dy = g'(x)dx$, and

$\int f(y)\,dy = \int f(g(x)\underbrace{g'(x))}\,dx$

don't forget this

Integration by parts:

$\int [f'(x)g(x) + f(x)g'(x)]\,dx = f(x)g(x)$ + constant

$[\int uv'\,dx = uv - \int u'v\,dx]$

→ Check out Paul's notes / Khan academy for more help.

- Textbook is not required
- Both sections will cover the same material
- Lab sessions utilized to cover additional examples.

Inner Product

You already saw "inner product" with vectors

$v = (v_1, \ldots, v_n)$; $u = (u_1, \ldots, u_n)$

$v \cdot u = v_1u_1 + v_2u_2 + \ldots + v_nu_n$

(standard Euclidean Inner Product)

- Functions can be considered as "infinite vectors":

  $f : [0, 1] \longrightarrow \mathbb{R}$ (Function) can be considered as "infinite vector" whose "components" are its values $f(x)$, $x \in [0, 1]$

- Inner product can be extended to functions:

  Given functions $f, g : D \to \mathbb{R}$

$$\boxed{f \cdot g = \int_D f(x)\, g(x)\, dx}$$

inner product between f, g

D is domain of f, g

For $f \cdot g$ to make sense, f and g must have the same domain. (Just like in the case: $v \cdot u$ NEEDS u and v to have the same number of components)

$\boxed{\text{Ex}}$ $f(x) = x$, $g(x) = x^2 + 1$
domain $[0, 1]$. Find $f \cdot g$

$f \cdot g = \int_0^1 f(x) \cdot g(x)\, dx$

$f \cdot g = \int_0^1 x^3 + x \; dx$

$\Rightarrow \left[x^4/4 + x^2/2 \Big|_0^1 \right] \longrightarrow \frac{1}{4} + \frac{1}{2} = 3/4$

Inner product depends on domain

Ex $f(x) = x$, $g(x) = x^2$, domain $[-1, 1]$

Find $f \cdot g$

$f \cdot g = \int_D f(x) \cdot g(x)\, dx$

$= \int_{-1}^{1} x^3\, dx \rightarrow \left[x^4/4 \Big|_{-1}^{1} \right]$

$\Rightarrow \quad 1/4 - 1/4 = \underline{0}$ → orthogonal

$\boxed{\text{2 functions } f, g \text{ are orthogonal when } f \cdot g = 0}$

From previous example: $f(x) = x$, $g(x) = x^2$ are orthogonal on $[-1, 1]$

But the same functions $f(x) = x$, $g(x) = x^2$ are not orthogonal on $[0, 1]$

$f \cdot g = \int_0^1 x^3\, dx = 1/4 \neq 0$

Orthogonality depends on the domain

Ex $f(x) = \cos x$, $g(x) = \sin^2 x$, domain $[0, \pi]$

Are f, g orthogonal ?

$f \cdot g = \int_0^{\pi} f(x)\, g(x)\, dx$

$= \int_0^{\pi} \cos x \sin^2 x\, dx = \frac{\sin^3 x}{3} \Big|_0^{\pi} = 0$

→ $f \perp g$ (f and g are orthogonal)

- A set of functions $\{f_1, f_2, \ldots, f_n, \ldots\}$ is an "orthogonal set" if all of its functions are mutually orthogonal

$f_k \perp f_j$ whenever $k \neq j$

Ex Given the function set

$\{ \sin(nx) : n = 1, 2, 3, \ldots \}$

Check if it is an orthogonal set

Take 2 arbitrary functions

→

$\sin(nx)$, $\sin(mx)$

and check if they are orthogonal

$\rightarrow$ $\sin(nx) \cdot \sin(mx)$

$= \int_0^{\pi} \sin(nx)\sin(mx)\,dx \qquad n \neq m$

$\cos(nx + mx) = \cos(nx)\cos(mx) - \sin(nx)\sin(mx)$

$\cos(nx - mx) = \cos(nx)\cos(mx) + \sin(nx)\sin(mx)$

$\Rightarrow \sin(nx)\sin(mx) = \dfrac{\cos(nx - mx) - \cos(nx + mx)}{2}$

So , $\int_0^{\pi} \sin(nx)\sin(mx)\,dx$

$= \frac{1}{2}\int_0^{\pi} \left(\cos[(n-m)x] - \cos[(n+m)x]\right) dx$

$= \frac{1}{2}\left[\dfrac{\sin[(n-m)x]}{n-m} - \dfrac{\sin[(n+m)x]}{n+m}\right]_0^{\pi} = 0$

$n \neq m \longrightarrow$ (non-zero) denominator & $n + m \neq 0$ as both are positive integers

$\Rightarrow \sin(nx) \perp \sin(mx)$ whenever $n \neq m$

$\Rightarrow$ $\boxed{\text{our set } \{\sin(nx) : n = 1, 2, 3, \dots\} \text{ is orthogonal set}}$ (*)

$\boxed{\textbf{Ex}}$ is $\{\cos(nx) : n = 1, 2, 3, \dots\}$ an orthogonal set? domain $[0, \pi]$

Take 2 arbitrary elements

$\cos(nx)$, $\cos(mx)$, $n \neq m$

Inner Product

$\int_0^{\pi} \cos(nx)\cos(mx)\,dx$

$\cos(nx)\cos(mx) = \dfrac{\cos(nx - mx) + \cos(nx + mx)}{2}$

$= \int_0^{\pi} \frac{1}{2}\left(\cos[(n-m)x] + \cos[(n+m)x]\right)dx$

$= \frac{1}{2}\left[\dfrac{\sin[(n-m)x]}{n-m} + \dfrac{\sin[(n+m)x]}{n+m}\right]_0^{\pi} = 0$

$\Rightarrow \cos(nx) \perp \cos(mx)$ whenever $n \neq m$

$\Rightarrow$ $\boxed{\{\cos(nx) : n = 1, 2, 3, \dots\} \text{ is orthogonal set}}$ (*)

- Weighted orthogonality:

$f \cdot g$ are orthogonal on domain D with respect to a <u>weight function ω</u> if:

$$\int_D f(x)g(x)\omega(x)\,dx = 0$$

Ex. Show $f(x) = 1$, $g(x) = 1 - x$ are orthogonal on domain $[0, +\infty]$ with weight function $\omega(x) = e^{-x}$

Have to check. $\int_0^{+\infty} f(x)g(x)\omega(x)dx = 0$:

$$\int_0^{+\infty} f(x)g(x)\omega(x)\,dx$$

$$= \int_0^{+\infty} 1 \cdot (1-x) \cdot e^{-x}\,dx = \int_0^{+\infty} (e^{-x} - xe^{-x})\,dx$$

$$= \int_0^{+\infty} e^{-x}\,dx - \int_0^{+\infty} xe^{-x}\,dx$$

$$\Rightarrow \cancel{\int_0^{+\infty} e^{-x}\,dx} - \underbrace{\left[\cancel{-xe^{-x}\Big|_0^{+\infty}} + \cancel{\int_0^{+\infty} e^{-x}\,dx}\right]}_{\text{obtained from integration by parts.}} = 0$$

<u>Norm</u>

vector $v = (v_1, \ldots, v_n)$ has norm

$$\|v\| = \sqrt{v_1^2 + \cdots + v_n^2} = \sqrt{v \cdot v}$$

- A function $f: D \to \mathbb{R}$ has norm

$$\|f\| = \sqrt{f \cdot f} = \sqrt{\int_D f(x)^2\,dx}$$

Again... Norm depends on the domain.

Ex. Find the norm of $f(x) = \sin(x)$ on $[0, \pi]$

$$\|f\| = \sqrt{\int_0^{\pi} \sin^2 x\,dx} = \sqrt{\int_0^{\pi} \frac{1-\cos(2x)}{2}\,dx}$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$1 = \cos^2 x + \sin^2 x$$

$$\Rightarrow \sin^2 x = \frac{1-\cos(2x)}{2} = \sqrt{\frac{\pi}{2} - \cancel{\sin(2x)}\Big|_0^{\pi}} = \sqrt{\pi/2}$$

- An orthogonal set (of functions) is "complete" if the <u>only</u> continuous function orthogonal to all functions in the set is $f(x) = 0$