OCT. 30/18

Ex. Change the order of iteration from

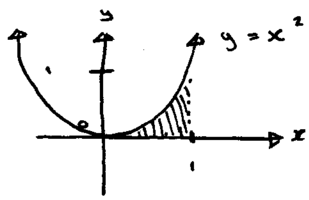
$$\int_0^1 \left[\int_0^{x^2} \left(\int_0^{y} f(x,y,z)\, dz \right) dy \right] dx$$

into: $dx\ dz\ dy$

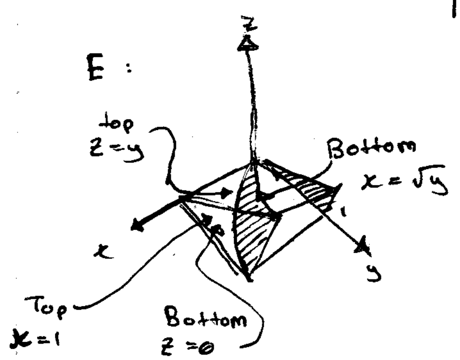
Solution: $\iiint_E f(x,y,z)\, dV$

$$E = \left\{ (x,y,z) : \begin{array}{l} 0 \le z \le y \\ 0 \le y \le x^2 \\ 0 \le x \le 1 \end{array} \right\}$$

In xy-plane:



E:



Now we want to look at E as follows:

$$E: \left\{ (x,y,z) : \begin{array}{l} \sqrt{y} \le x \le 1 \\ 0 \le y \le 1 \\ 0 \le z \le y \end{array} \right\}$$

$$\Rightarrow \iiint_E f(x,y,z)\, dV = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\, dx\, dz\, dy$$

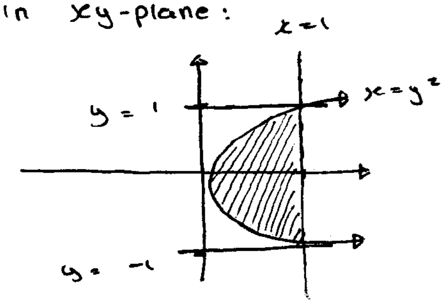
Ex Change the order from

$$\int_{-1}^1 \int_{y^2}^1 \int_0^{1-x} f(x,y,z)\, dz\, dx\, dy \quad : \quad \text{into } dx\ dy\ dz$$

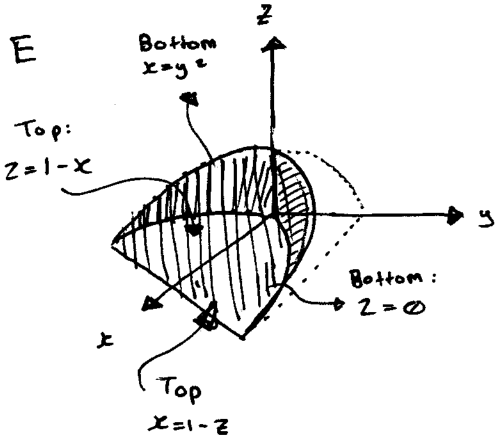
Solution $\iiint_E f(x,y,z)\, dV$

$$E = \left\{ (x,y,z) : \begin{array}{l} 0 \le z \le 1-x \\ y^2 \le x \le 1 \\ -1 \le y \le 1 \end{array} \right\}$$

In xy-plane:



E

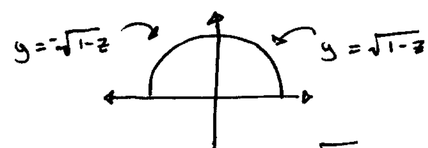


Now we want to write as follows:

$$E = \left\{ (x,y,z) : \begin{array}{c} \overset{\text{BOTTOM}}{y^2} \le x \le \overset{\text{TOP}}{1-z} \\ -\sqrt{1-z} \le y \le \sqrt{1-z} \\ 0 \le z \le 1 \end{array} \right\}$$

Intersection

$y^2 = 1-z \;:\; z = 1-y^2$



$y^2 = 1-z$

$y = \pm\sqrt{1-z}$

Solution: $\displaystyle\iiint_E f(x,y,z)\,dV = \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{y^2}^{1-z} f(x,y,z)\,dx\,dy\,dz$

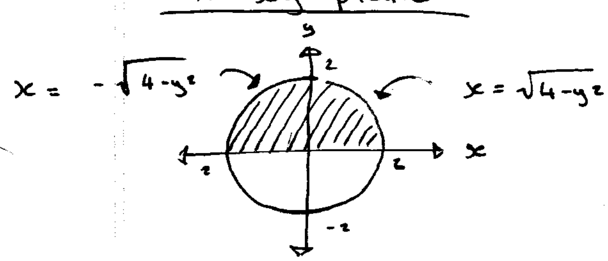
Ex: Evaluate the iterated integrals by changing to cylindrical coordinates:

$$\int_0^2 \left[\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \left(\int_{\sqrt{x^2+y^2}}^{2} xz\,dz \right) dx \right] dy$$

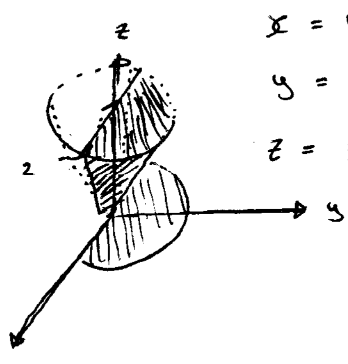
Solution: $\displaystyle = \iiint_E xz\,dV$

$$E = \left\{ (x,y,z) : \begin{array}{c} \overset{\text{CONE}}{\sqrt{x^2+y^2}} \le z \le \overset{\text{PLANE}}{2} \\ -\sqrt{4-y^2} \le x \le \sqrt{4-y^2} \\ 0 \le y \le 2 \end{array} \right\}$$

In xy-plane:



E:



$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Domain in xy-plane $= \left\{ (r, \theta) = \begin{array}{l} 0 \le \theta \le \pi \\ 0 \le r \le 2 \end{array} \right\}$

$$E = \left\{ (z, r, \theta) : \begin{array}{l} r \le z \le 2 \\ 0 \le \theta \le \pi \\ 0 \le r \le 2 \end{array} \right\}$$

$$= \iiint_E xz\, dV = \int_0^{\pi} \int_0^{2} \int_r^{2} (r\cos\theta \; z)\underbrace{(r)}_{\text{extra term}}\, dz\, dr\, d\theta$$

$$\Rightarrow \int_0^{\pi} \int_0^{2} \left(\int_r^{2} r^2 \cos\theta \; z\, dz \right) dr\, d\theta$$

$$\to \int_0^{\pi} \int_0^{2} r^2 \cos\theta \left. \frac{z^2}{2} \right|_{z=r}^{z=2} dr\, d\theta$$

$$\to \int_0^{\pi} \int_0^{2} \left(2r^2 \cos\theta - \tfrac{1}{2} r^4 \cos\theta \right) dr\, d\theta$$

$$\to \int_0^{\pi} \left(2 r^3/3 \cos\theta - \tfrac{1}{2} \cdot r^5/5 \cos\theta \right) \Big|_{r=0}^{r=2} d\theta$$

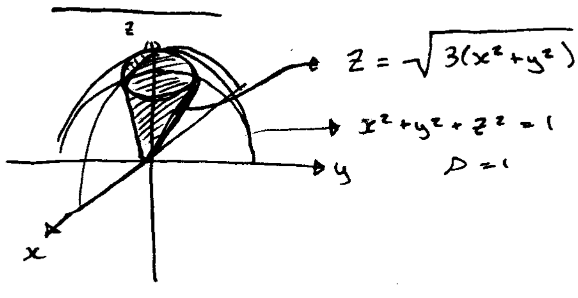
$$\to \int_0^{\pi} \left(16/3 \cos\theta - \tfrac{16}{5} \cos\theta \right) d\theta \Rightarrow \int_0^{\pi} \tfrac{32}{15} \cos\theta\, d\theta$$

$$\Rightarrow 32/15 \sin\theta \Big|_{\theta=0}^{\theta=\pi} = 0$$

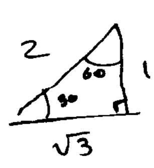
Triple Integrals in Spherical Coordinates

Ex. Let E be the solid bounded by the cone $z = \sqrt{3(x^2+y^2)}$ and the sphere $x^2+y^2+z^2 = 1$. Write E using spherical coordinates.

Solution :



$$E = \left\{ (\rho, \theta, \phi) : \begin{array}{l} 0 \le \rho \le 1 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi/6 \end{array} \right\}$$



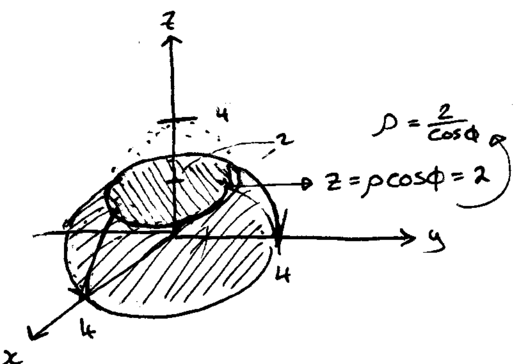
Triple Integrals in Spherical Coord.

$$\iiint_E f(x,y,z)\, dV$$

where $E = \left\{ (\rho, \phi, \theta) : \begin{array}{l} a \le \rho \le b \\ c \le \phi \le d \\ e \le \theta \le f \end{array} \right\}$

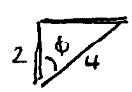
$$\int_e^f \int_c^d \int_a^b f(\underbrace{\rho \sin\phi \cos\theta}_{x}, \underbrace{\rho \sin\phi \sin\theta}_{y}, \underbrace{\rho\cos\phi}_{z}) \underbrace{\rho^2 \sin\phi}_{\text{EXTRA TERM}}\, d\rho\, d\phi\, d\theta$$

Ex. Compute using spherical coordinates, the volume of the solid inside the sphere $x^2+y^2+z^2 = 16$ and bounded by $z = 0$ and $z = 2$



$E \{ (\rho, \phi, \theta) :$

$0 \le \rho \le 4$	$0 \le \rho \le \frac{2}{\cos\phi}$
$0 \le \theta \le 2\pi$	$0 \le \theta \le 2\pi$
$\pi/3 \le \phi \le \pi/2$	$0 \le \phi \le \pi/3$



$\cos\phi = 1/2$

$$\boxed{\phi = \pi/3}$$

$\mathrm{Vol}(E) = \iiint_E 1\, dV$ Spherical coord

extra term

$$\Rightarrow \int_{\pi/3}^{\pi/2} \int_0^{2\pi} \int_0^4 1 \sin^2\phi\, d\rho\, d\theta\, d\phi \ \ldots$$

$$\ldots + \int_0^{\pi/3} \int_0^{2\pi} \int_0^{2/\cos\phi} 1 \sin^2\phi\, d\rho\, d\theta\, d\phi$$

Computation 2 : $\iint \left(\left(\rho^3/3\right) \sin\phi \right) \Big|_{\rho = 0}^{\rho = 2/\cos\phi} d\theta\, d\phi$

$$\Rightarrow \int_0^{\pi/3} \int_0^{2\pi} \left(8/3 \frac{1}{\cos^3\phi} \cdot \sin\phi \right) d\theta\, d\phi \rightarrow \int_0^{\pi/3} \left(\frac{16\pi}{3}\right)\left(\frac{1}{\cos^3\phi}\right)(\sin\phi)\, d\phi$$

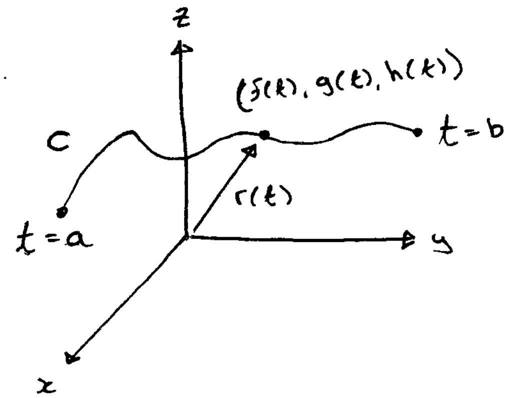
$u = \cos\phi$

$du = -\sin\phi\, d\phi$

$-du = \sin\phi\, d\phi$

$$\int_1^{1/2} \frac{16\pi}{3} u^{-3}\,(du) \longrightarrow \frac{16\pi}{3}\left(-\frac{u^{-2}}{2}\right)\Bigg|_{u=1/2}^{u=1} = \ldots$$

Chapter 15 - Line and Surface Integrals



Goal : to define

$$\int \underbrace{f(x, y, z)}_{C}$$

Preparation: $x = f(t)$

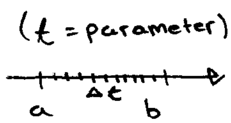
$y = g(t)$

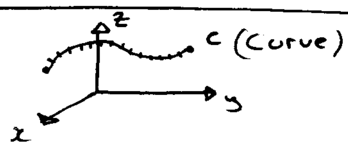
$z = h(t)$

$a \le t \le b$

$$r(t) = \langle f(t), g(t), h(t) \rangle$$
$$= f(t)\,i + g(t)\,j + h(t)\,k$$

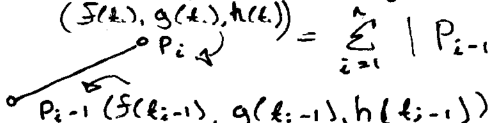
Computation of the arc length of C





$t_0 < t_1 < t_2 < t_n$

arc length $\approx \sum_{i=1}^{n}$ length of i^{th} piece



$$= \sum_{i=1}^{n} |P_{i-1} P_i| \Rightarrow \sum_{i=1}^{n} \sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2 \cdots \cdots [h(t_i) - h(t_{i-1})]^2}$$

$$\longrightarrow \sum_{i=1}^{n} \sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2 + [h(t_i) - h(t_{i-1})]^2}$$

Calculus I:

Cauchy Thm. $f(b) - f(a) = f'(c)(b-a)$

(between a and b)

$$\text{arc length (of } C) = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^{*})\Delta t]^2 + [g'(t_i^{**})\Delta t]^2 + [h'(t_i^{***})\Delta t]^2}$$

(between t_{i-1} and t_i) (between t_{i-1} and t_i)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^{*})^2 + (g'(t_i^{**}))^2 + (h'(t_i^{***}))^2]}\,\Delta t$$

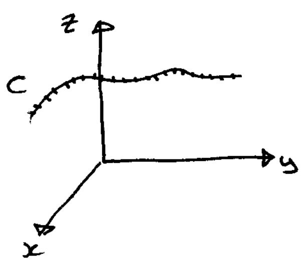
$$= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\; dt$$

$$\Rightarrow \int_a^b \|r'(t)\|\, dt$$

where $r(t) = \langle f(t), g(t), h(t) \rangle$

$r'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Notation (From now on):



$C:\quad x = x(t)$

$y = y(t)$

$z = z(t)$

$a \le t \le b$

Definition of a line integral

Setting: Assume that a thin wire C has density $f(x, y, z)$ at every point (x, y, z) on C.

What is the total mass?

Answer:

$$\text{Total mass} = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \text{mass of } i^{th} \text{ piece}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(\underbrace{x(t_i), y(t_i), z(t_i)}_{\text{at time } t}) \sim \text{arc length } i^{th} \text{ piece}$$

$$= \int_a^b \underbrace{f(r(t))}_{\substack{\text{density}\\ \text{part}}} \underbrace{\|r'(t)\|\, dt}_{\substack{\text{arc length}\\ \text{part}}}$$

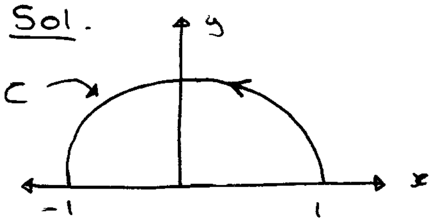
DEF'N: "$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \overbrace{\|r'(t)\|}^{\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}}\, dt$

Line integral of f along C w.r.t. arc length.

Ex. Evaluate $\int_c (2 + x^2 y)\, ds$

where C is a 2-dimensional curve representing the upper half of the unit area.

Sol.



$x = 1 \cdot \cos t = \cos t$

$y = 1 \sin t = \sin t$

$0 \le t \le \pi$

$r(t) = \langle \cos t, \sin t \rangle$

$r'(t) = \langle -\sin t, \cos t \rangle$

$$\|r'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2}$$
$$= \sqrt{1} = 1$$

Now $$\int_c (\underbrace{2 + x^2 y}_{(x,y)})\, ds = \int_0^{\pi} \underbrace{2 + (\cos t)^2 (\sin t)}_{f(r(t))}\ \underbrace{(1)}_{\|r'(t)\|}\, dt$$

$\ast$

$$\Rightarrow \int_0^{\pi} \left[2 + (\cos t)^2 \sin t \right] dt = 2t - \frac{(\cos t)^3}{3} \Bigg|_{t=0}^{t=\pi}$$

$$\Rightarrow 2\pi - \frac{(\cos \pi)^3}{3} - \left(- \frac{\cos(0))^3}{3} \right)$$

$$\Rightarrow 2\pi + 1/3 + 1/3 = 2\pi + 2/3$$