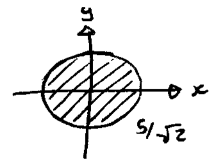
Oct. 23/18

LAST TIME:

$$V = \iint_D \left(\sqrt{25-(x^2+y^2)} - \sqrt{x^2+y^2} \right) dA$$

D = disk of radius $5/\sqrt{2}$ in x-y plane



$$\rightarrow \left\{ (r,\theta) \quad \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le r \le 5/\sqrt{2} \end{matrix} \right\}$$

where $x = r\cos\theta$

$y = r\sin\theta$

extra term

$$\rightarrow \int_0^{2\pi}\int_0^{5/\sqrt{2}} \left(\sqrt{25-r^2} - \underbrace{\sqrt{r^2}}_{(=r)} \right) r \; dr\, d\theta$$

$$\rightarrow \int_0^{2\pi} \left[\int_0^{5/\sqrt{2}} r\sqrt{25-r^2}\, dr - \int_0^{5/\sqrt{2}} r^2 dr \right] d\theta$$

$\rightarrow$ $u = 25-r^2$

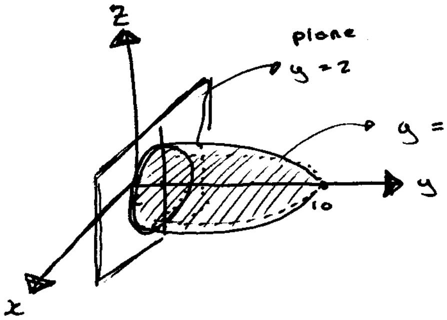
$du = -2r\, dr$

$dr = \left(\frac{-1}{2r}\right) du$

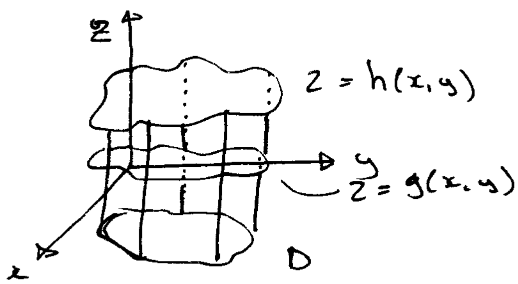
$$\rightarrow \int_0^{2\pi} \left[\int_{25}^{25/2} \cancel{r} \sqrt{u} \left(\frac{-1}{2\cancel{r}}\right) du - r^3/3 \Big|_{r=0}^{r=5/\sqrt{2}} \right] d\theta \; \underline{\underline{\ldots \text{etc.}}}$$

Ex: Find the volume of the solid bounded by the paraboloid $y = 10 - 3x^2 - 3z^2$ and plane $y = 2$

Sol:



where



$$\text{volume} = \iint_D h(x,y)\,dA \cdots$$
$$\cdots - \iint_D g(x,y)\,dA$$

v_1 v_2

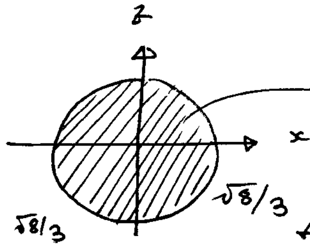
then, $$\text{Volume } \iint_D (10 - 3x^2 - 3z^2)\,dA - \iint_D 2\,dA$$

$$\Rightarrow \iint_D (8 - 3x^2 - 3z^2)\,dA$$

Information about **D**: intersection between $\begin{cases} y = 10 - 3x^2 - 3z^2 \\ y = 2 \end{cases}$

$$\rightarrow 10 - 3x^2 - 3z^2 = 2$$
$$x^2 + z^2 = 8/3$$

Now:



$$D = (r, \theta) = \begin{Bmatrix} 0 \le \theta \le 2\pi \\ 0 \le r \le \sqrt{8/3} \end{Bmatrix}$$

extra term

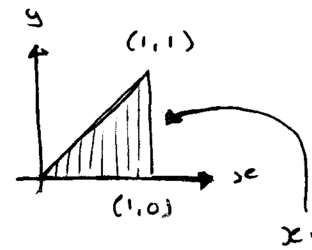
Now: $$\int_0^{2\pi} \int_0^{\sqrt{8/3}} (8 - 3r^2) \cdot r \, dr\, d\theta \Rightarrow \int_0^{2\pi} \int_0^{\sqrt{8/3}} 8r - 3r^4 \, dr\, d\theta$$

... etc.

Ex. Compute, using polar coordinates:

$$\iint_D \frac{1}{(1+x^2+y^2)^{3/2}}\, dA$$

where D = region in x-y plane:



Solution

$$D = \left\{ (r,\theta) : \begin{array}{l} 0 \le \theta \le \pi/4 \\ 0 \le r \le \frac{1}{\cos\theta} \end{array} \right\}$$

then $r\cos\theta = 1$

$r = \frac{1}{\cos\theta}$

$$\iint_D \frac{1}{(1+x^2+y^2)^{3/2}}\, dA \underset{\substack{\uparrow \\ \text{POLAR COORD}}}{=} \int_0^{\pi/4} \left[\int_0^{1/\cos\theta} \frac{1}{(1+r^2)^{3/2}} \cdot \underbrace{r}_{\text{extra term}}\, dr \right] d\theta$$

$$\Rightarrow \int_0^{\pi/4} \left[\int_0^{1/\cos\theta} \frac{r}{(1+r^2)^{3/2}}\, dr \right] d\theta \underset{\text{SUBST}}{=} \int_0^{\pi/4} \left[\int_1^{1+\frac{1}{\cos^2\theta}} u^{-3/2} \cdot \tfrac{1}{2}\, du \right] d\theta$$

$u = 1 + r^2$
$du = 2r\,dr$
$dr = \frac{1}{2r}\,du$

$r = 0 \rightsquigarrow u = 1$
$r = \frac{1}{\cos\theta} \rightsquigarrow u = 1 + \left(\frac{1}{\cos^2\theta}\right)$

$$\Rightarrow \int_0^{\pi/4} (1/2) \cdot \frac{u^{-1/2}}{(-1/2)} \Bigg|_{u=1}^{u = 1+\frac{1}{\cos^2\theta}} d\theta$$

$$\Rightarrow \int_0^{\pi/4} \left[\frac{-(1)}{\left(1 + \frac{1}{\cos^2\theta}\right)^{1/2}} - (-1) \right] d\theta = \int_0^{\pi/4} \left[1 - \frac{1}{\left(1+\frac{1}{\cos^2\theta}\right)} \right] d\theta$$

$$\Rightarrow \int_0^{\pi/4} 1\, d\theta - \int_0^{\pi/4} \frac{\cos\theta}{(1+\cos^2\theta)^{1/2}}\, d\theta = \theta \Big|_{\theta=0}^{\theta=\pi/4} - \int_0^{\pi/4} \frac{\cos\theta}{(2-\sin^2\theta)^{1/2}}\, d\theta$$

Substitute: $w = \sin\theta$

$dw = \cos\theta\, d\theta$

$$\Rightarrow \pi/4 - \int \frac{1}{\sqrt{2-w^2}}\, dw$$

$$\Rightarrow \pi/4 - \arcsin\left(w/\sqrt{2}\right)\Big|_{w=0}^{w=1/\sqrt{2}} = \pi/4 - \arcsin\left(\frac{1/\sqrt{2}}{\sqrt{2}}\right)$$

$$= \pi/4 - \arcsin 1/2 = \pi/4 - \pi/6 = \pi/12$$

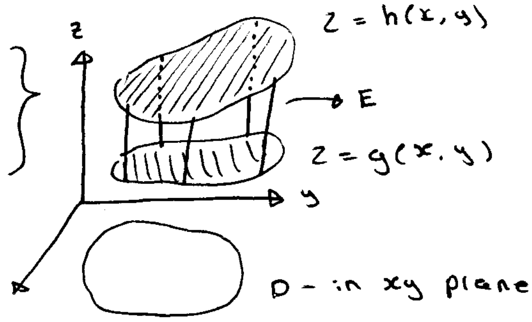
Triple Integrals

$$\iiint_E f(x,y,z)\,dV$$

E = Solid in 3-dimensions

Type-I domain E

$$E = \left\{ \begin{array}{l} (x,y,z) \;:\; g(x,y) \le z \le h(x,y) \\ \quad \text{with } (x,y) \text{ in } D = \text{domain} \\ \quad\quad \text{in } xy\text{-plane} \end{array} \right\}$$

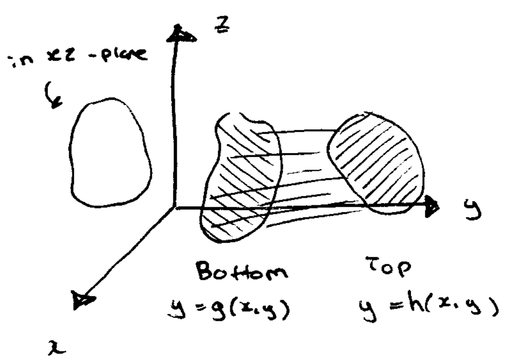


We will compute the triple integral as:

$$\iiint f(x,y,z)\,dV = \iint \left[\int_{g(x,y)}^{h(x,y)} f(\underline{\underline{x}},\underline{\underline{y}},z)\,dz \right] dA$$

Fixed

Type-II domain E



$$E = \left\{ \begin{array}{l} (x,y,z) \;:\; g(x,z) \le y \le h(x,z) \\ \quad \text{with } (x,z) \text{ in } D\text{-in } xz \text{ plane} \end{array} \right\}$$

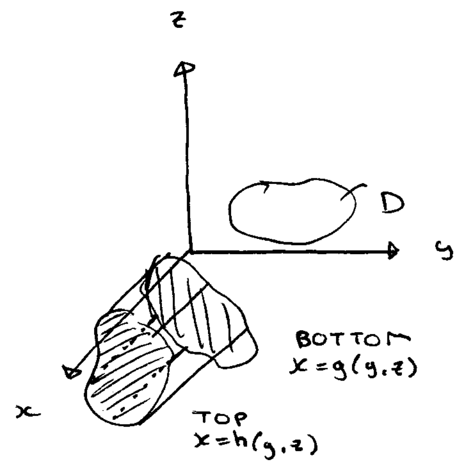
We will compute the integral as:

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{g(x,z)}^{h(x,z)} f(\underline{\underline{x}},y,\underline{\underline{z}})\,dy \right] dA$$

Fixed

Type III domain E

$$E = \left\{ \begin{array}{c} (x, y, z) : g(y,z) \le x \le h(y,z) \\ \text{with } (y, z) \text{ in } D \\ \text{in } yz\text{-plane} \end{array} \right\}$$

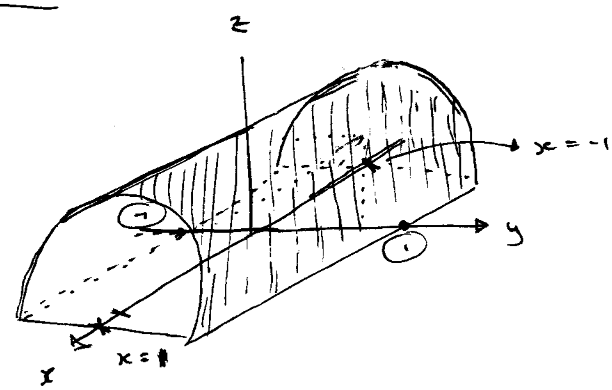


Triple Integrals

Ex: Compute $\iiint_E \underbrace{x^2 e^y z}_{f(x,y,z)} \, dV$

where E is the solid bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $z = 0$, $x = 1$, $x = -1$

Solution: E:



Bottom Top

$E = \{(x, y, z) : 0 \leq z \leq 1 - y^2\}$

Intersection: $z = 1 - y^2$
$z = 0$

$y^2 - 1 = 0$

$y^2 = 1 \rightarrow y = 1, y = -1$
(line) (line)

$$\iiint_E x^2 e^y z \, dV = \iint_D \left(\int_0^{1-y^2} x^2 e^y z \, dz \right) dA$$

$$= \iint_D x^2 e^y \left(z^2/2 \Big|_0^{1-y^2} \right) dA$$

$$\Rightarrow dA = \iint_D \frac{x^2 e^y (1-y)^2}{2} \, dA$$

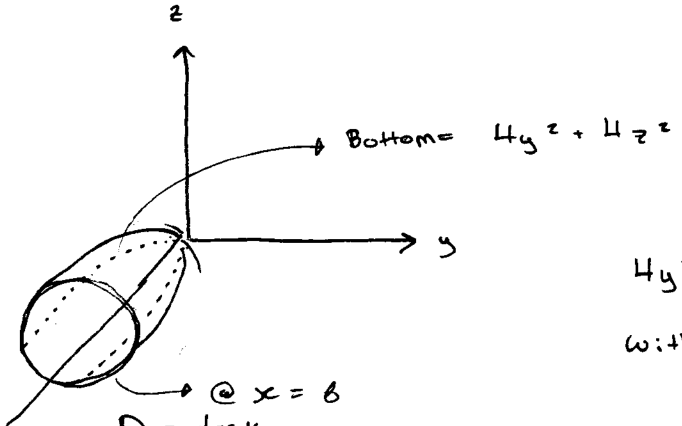
$$\Rightarrow \int_{-1}^{1} \left(\int_{-1}^{1} x^2 e^y \frac{(1-y^2)^2}{2} \, dx \right) dy = \int_{-1}^{1} \frac{x^3}{3} e^y \frac{(1-y^2)^2}{2} \Bigg|_{x=-1}^{x=1} dy$$

$$\Rightarrow \int_{-1}^{1} \frac{2}{3} e^y \frac{(1 - 2y^2 + y^4)}{2} \, dy = \ldots$$

Ex. Compute $\iiint_E \underbrace{\sqrt{y^2+z^2}}_{f(x,y,z)}\, dV$

where E is the solid bound by the paraboloid $x = 4y^2 + 4z^2$ and plane $x = 6$

Sol:



$4y^2 + 4z^2 \le x \le 6$

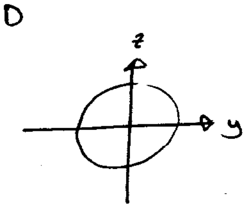
with y, z in $D =$

D = disk

$$\left.\begin{aligned} 4y^2+4z^2 &= x \\ x &= 6 \end{aligned}\right\} \quad \begin{aligned} 6 &= 4y^2+4z^2 \\ 6/4 &= y^2+z^2 \\ (\sqrt{3}/2)^2 &= y^2+z^2 \end{aligned}$$

$$\iiint_E \sqrt{y^2+z^2}\, dV = \iint_D \left(\int_{4(y^2+z^2)}^{6} \sqrt{y^2+z^2}\, dx \right) dA = \iint_D x\sqrt{y^2+z^2} \Big|_{x=4(y^2+z^2)}^{x=6} dA$$

$$\Rightarrow \iint_D 6\sqrt{y^2+z^2} - 4(y^2+z^2)(\sqrt{y^2+z^2})\, dA$$



$$\Rightarrow \int_0^{2\pi} \left[\int_0^{\sqrt{3}/2} (6r - 4r^2 \cdot r) \cdot \overset{\text{extra term}}{r}\, dr \right]$$

$$\left.\begin{aligned} y &= r\cos\theta \\ z &= r\sin\theta \end{aligned}\right\} \quad y^2+z^2 = r^2$$

$$\Rightarrow \int_0^{2\pi} \left[\int_0^{\sqrt{3}/2} (6r^2 - 4r^4)\, dr \right] d\theta$$

Remark: We solved this problem by using a change of variables to cylindrical coordinates, in this case: $\begin{cases} x = x \\ y = r\cos\theta \\ z = r\sin\theta \end{cases}$

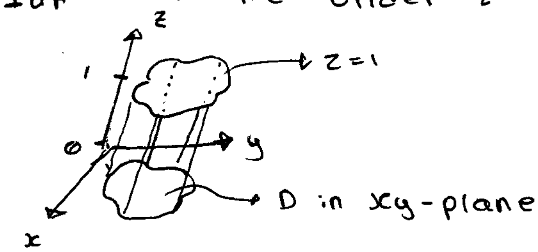
In general, $\exists$

Remark: $\iiint_E \underbrace{f(x,y,z)}_{\text{positive}}\, dV$ represents geometrically computations of 4 dim. volume.

However: $\iiint_E 1\, dV = \text{Vol}(E)$

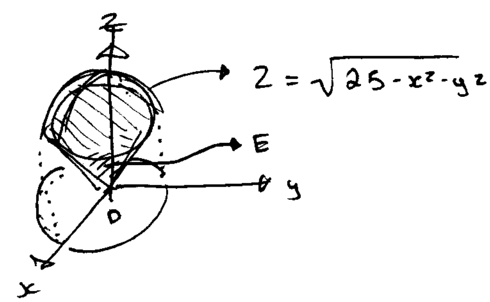
Case: double integral

$\iint_D 1\, dA$ = volume under $z=1$



$= \underset{\text{height}}{1} \times \text{area}(D) = \text{area of } D$

Ex:



Sol #1

$$V = \iint_D \sqrt{25-x^2-y^2}\, dA = \iint_D \sqrt{x^2+y^2}\, dA$$

$$\Rightarrow \iint_D \left(\sqrt{25-x^2-y^2} - \sqrt{x^2+y^2}\right) dA$$

Sol #2 Volume $(E) = \iiint_E 1\, dV$

$$E = \left\{ (x,y,z) : \overset{\text{top}}{\sqrt{x^2+y^2}} \le z \le \overset{\text{bottom}}{\sqrt{25-x^2-y^2}} \text{ with } (x,y) \text{ in } D = \text{disc in } xy\text{-plane} \right\}$$

$$\Rightarrow \iint_D \left(\int_{\sqrt{x^2+y^2}}^{\sqrt{25-x^2-y^2}} dz \right) dA = \iint_D \left(z \Big|_{z=\sqrt{x^2+y^2}}^{z=\sqrt{25-x^2-y^2}} \right) dA$$

$$\Rightarrow \iint_D \left(\sqrt{25-x^2-y^2} - \sqrt{x^2+y^2} \right) dA$$