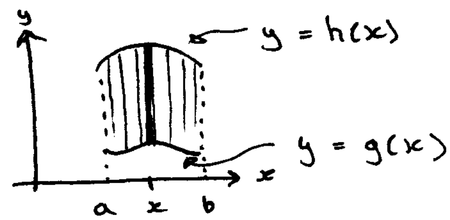
Oct. 16/18

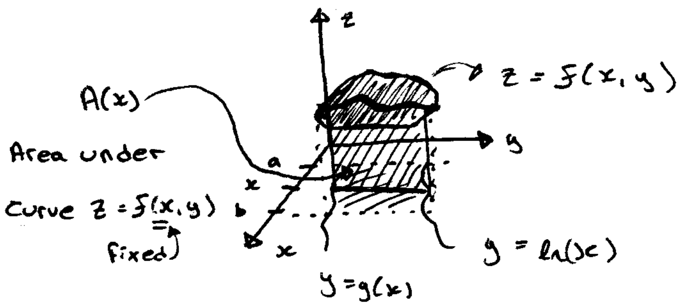
Double integrals For general domains

$\iint_D f(x,y)\,dA$ where D = domain in x-y plane

1. Type I domain **D**

$$D = \{(x,y) : a \le x \le b,\ g(x) \le y \le h(x)\}$$



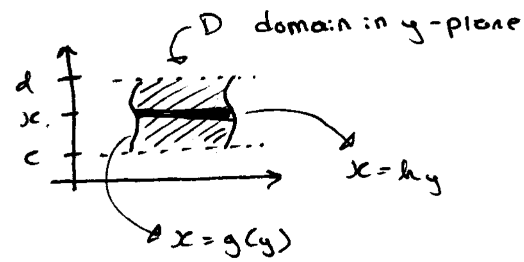


$$\iint_D f(x,y)\,dA = \int_a^b A(x)\,dx$$

$$\Rightarrow \int_a^b \left[\int_{g(x)}^{f(x)} f(\underset{\text{Fixed}}{x}, y)\,dy\right] dx$$

1. Type II domain D

$$D = \{(x,y) : c \le y \le d,\ g(y) \le x \le h(y)\}$$



We will get:

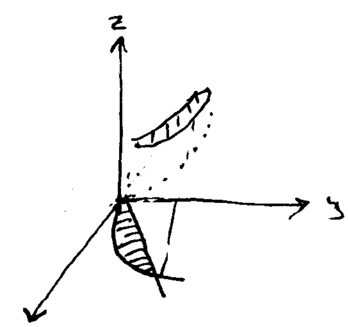
$$\iint_D f(x,y)\,dA = \int_c^d \left[\int_{g(y)}^{h(y)} f(x,\underset{\text{Fixed}}{y})\,dx\right] dy$$

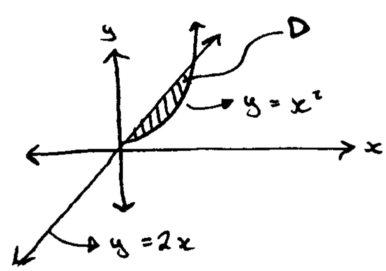
Example

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above D, which is the region in xy-plane bounded by $y = 2x$ and $y = x^2$

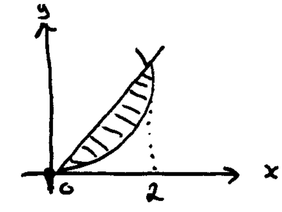
Solution

$$\text{Volume} = \iint_D \underbrace{x^2 + y^2}_{f(x,y)} \qquad D: \text{ in } x\text{-}y \text{ plane}$$





D : in x-y plane



Intersection:

$$\left.\begin{array}{l} y = 2x \\ y = x^2 \end{array}\right\} \quad \begin{array}{l} x^2 = 2x \\ x^2 - 2x = 0 \\ x(x-2) = 0 \end{array}$$

$$D = \left\{ (x,y) : \begin{array}{l} 0 \le x \le 2 \\ x^2 \le y \le 2 \end{array} \right\}$$

type I

Now: $\text{Volume} = \iint_D f(x,y)\, dA = \int_0^2 \left[\int_{x^2}^{2x} (\underbrace{x^2}_{\text{Fixed}} + y^2)\, dy \right] dx$

$$\Rightarrow \int_0^2 \left[\left(x^2 y + y^3/3 \right) \Big|_{y=x^2}^{y=2x} \right] dx = \int_0^2 \left(\left(2x^3 + 8x^3/3 \right) - \left(x^4 + x^6/3 \right) \right)$$

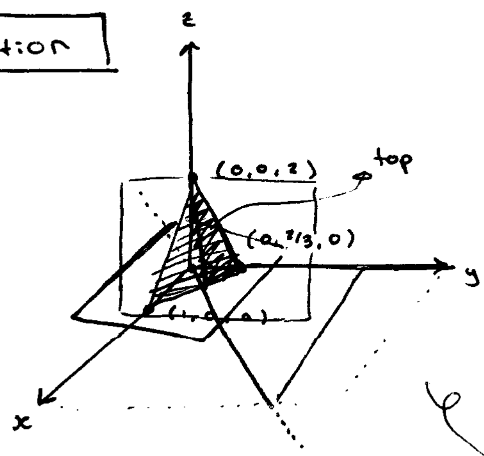
$$= \int_0^2 \left(14/3\, x^3 - x^4 - x^6/3 \right) dx$$

Example

Use double integrals to compute the volume of the solid bounded by the planes

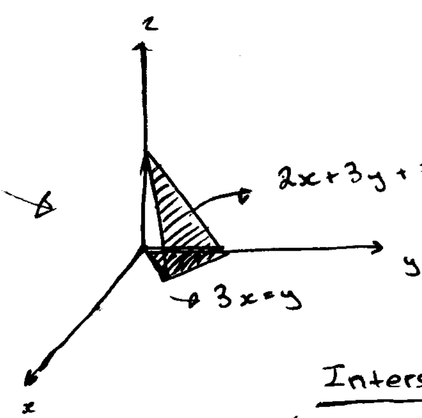
$$2x + 3y + z = 2$$
$$3x = y$$
$$x = 0$$
$$z = 0$$

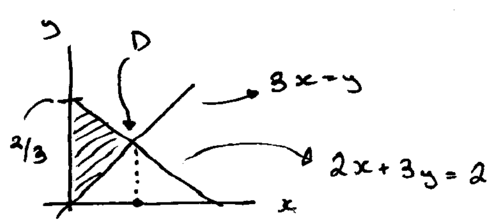
Solution



$\text{Volume} = \iint (2 - 2x - 3y)\, dA$

Top : $2x + 3y + z = 2$





Intersection

$$\begin{cases} y = 3x \\ 2x + 3y = 2 \end{cases} \rightsquigarrow 2x + 3(3x) = 2$$

$$11x = 2$$
$$x = 2/11$$

$$D = \left\{ (x,y) \quad \begin{array}{l} 0 \le x \le \overset{2/11}{?} \\ 3x \le y \le \frac{2-2x}{3} \end{array} \right\}$$

type I domain

$$2x + 3y = 2 \rightsquigarrow y = \frac{2-2x}{3}$$

Now:

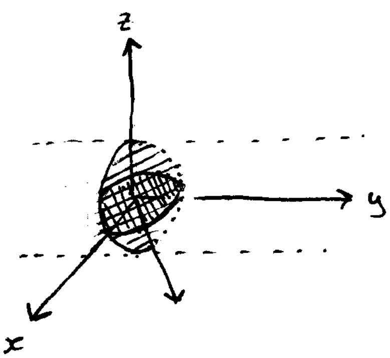
$$\text{Volume} = \iint_D (2-2x-3y)\,dA$$

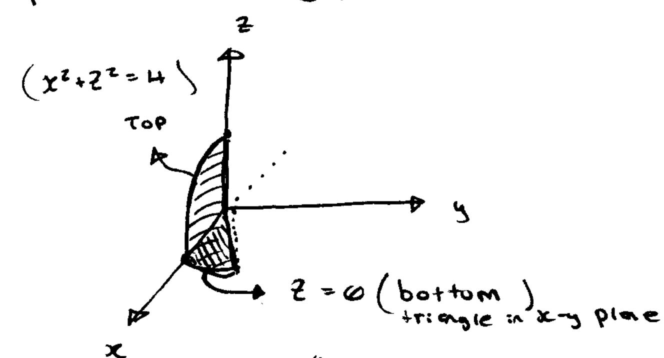
$$\Rightarrow \int_0^{2/11}\left[\int_{3x}^{\frac{2-2x}{3}} (2-2x-3y)\,dy\right]dx$$

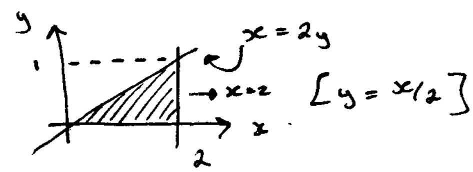
$$\Rightarrow \int_0^{2/11} \left(2y-2xy-3y^2/2\right)\Bigg|_{y=3x}^{y=\frac{2-2x}{2}} dx$$

$$\Rightarrow \int_0^{2/11} \left(2\left(\frac{2-2x}{3}\right) - 2x\cdot\frac{2-2x}{3} - \frac{3}{2}\left(\frac{2-2x}{3}\right)^2\right) - \cdots \text{ etc.}$$

Example Set up, but do not evaluate, the iterated integrals for the computation of the volume of the solid bounded by the cylinder $x^2+z^2=4$ and the planes $x=2y$, $y=0$, $z=0$ (in the first octet)







$$\Rightarrow \iint_D \underbrace{\sqrt{4-x^2}}_{f(x,y)}\,dA$$

We can write D as follows:

① as type I

$$D=\{(x,y):\ 0\le x\le 2,\ 0\le y\le x/2\}$$

② as type II

$$D=\{(x,y):\ 0\le y\le 1,\ 2y\le x\le 2\}$$

with ①: $= \displaystyle\int_0^2\left[\int_0^{x/2}\sqrt{4-x^2}\,dy\right]dx$

②: $\displaystyle\int_0^1\left[\int_{2y}^2\sqrt{4-x^2}\,dx\right]dy$

$\boxed{\text{Ex:}}$ Evaluate the integrated integrals

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$\int_0^1 \left(\int_{2y}^3 e^{x^2}\, dx \right) dy$

by changing the order of integration / iteration

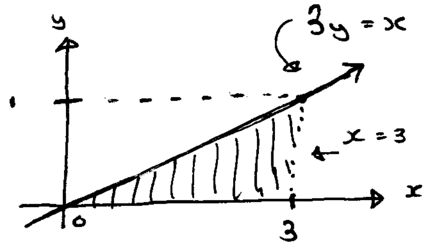
$\boxed{\underline{\text{Solution}}:}$ Remark: this is $\underline{\text{not}}$ the way to do it

$$\int_0^1 \left(\int_{3y}^3 e^{x^2}\, dx \right) dy \neq \int_{3y}^3 \left(\int_0^1 e^{x^2}\, dy \right) dx$$

The correct way:

$$\int_0^1 \left(\int_{3y}^3 e^{x^2} dx \right) dy = \iint_D e^{x^2}\, dA$$

where $D = \left\{ (x, y) = \begin{matrix} 0 \leq y \leq 1 \\ 3y \leq x \leq 3 \end{matrix} \right\}$



Same D:

$$D = \left\{ (x, y) : \begin{matrix} 0 \leq x \leq 3 \\ 0 \leq y \leq x/3 \end{matrix} \right\}$$

$$\Rightarrow \int_0^1 \left(\int_{3y}^3 e^{x^2}\, dx \right) dy = \int_0^3 \left[\int_0^{x/3} e^{x^2}\, dy \right] dx$$

$$\Rightarrow \int_0^3 e^{x^2} y \Big|_{y=0}^{y=x/3} dx = \int_0^3 e^{x^2} \frac{x}{3}\, dx = \int_0^9 e^u \, {}^1\!/_3 \cdot {}^1\!/_2\, du$$

Subst: $u = x^2$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$$= \frac{1}{6} e^u \Big|_{u=0}^{u=9} = \frac{1}{6}(e^9 - 1)$$

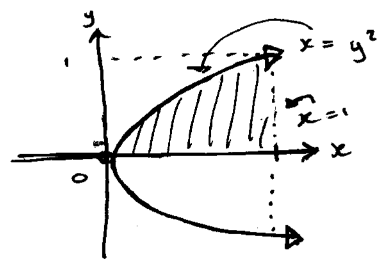
$\boxed{\text{Example :}}$ Compute $\int_0^1 \left(\int_{y^2}^1 y \sin(x^2)\, dx \right) dy$

Solution :

$$\int_0^1 \left(\int_{y^2}^1 y \sin(x^2)\, dx \right) dy = \iint_D y \sin(x^2)\, dA$$

$$D = \left\{ (x, y) \Rightarrow \begin{matrix} 0 \le y \le 1 \\ y^2 \le x \le 1 \end{matrix} \right\}$$

type II
(y - First)



$x = y^2 \rightsquigarrow y = \sqrt{x}$

$\to$ Switching ...

$$\text{New } D = \left\{ (x, y) \Rightarrow \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le \sqrt{x} \end{matrix} \right\}$$

$$\int_0^1 \left(\int_0^{\sqrt{x}} y \sin(x^2)\, dy \right) dx$$

$$\Rightarrow \int_0^1 \frac{y^2}{2} \sin(x^2) \Big|_{y=0}^{y=\sqrt{x}} dx \Rightarrow \int_0^1 \frac{x}{2} \sin(x^2)\, dx$$

$$\Rightarrow \int_0^1 \frac{\cancel{x}}{2} \sin(u) \frac{du}{2\cancel{x}}$$

where $u = x^2$

$du = 2x\, dx$

$dx = du/2x$

$$\Rightarrow \frac{1}{4} \int_0^1 \sin(u)\, du$$

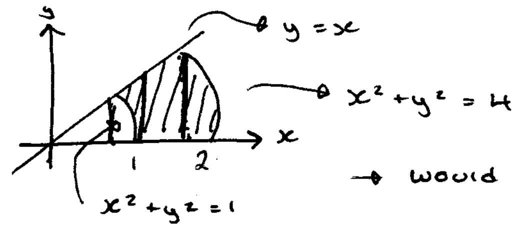
$$\Rightarrow \frac{1}{4} \left(-\cos(u) \Big|_{u=0}^{u=1} \right) \Rightarrow \left(\frac{1}{4}\right)(-\cos(1) + 1)$$

$$\boxed{\Rightarrow \left(\frac{1}{4}\right) - \frac{\cos(1)}{4}}$$

Question: how do we compute:

$$\iint \arctan(y/x)\, dA$$

where D is:



→ would have to compute 3 regions

__Remark:__

Same D: (but in polar coordinates)

$$D = \left\{ (r, \theta) \Rightarrow \begin{array}{l} 0 \le \theta \le \pi/4 \\ 1 \le r \le 2 \end{array} \right\} \text{ (just a rectangle)}$$

→ __Change of Variables to Polar Coordinates__

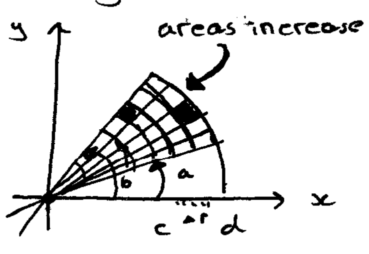
$$\iint_D f(x,y)\, dA$$

when $D \left\{ (r,\theta) \Rightarrow \begin{array}{l} a \le \theta \le b \\ c \le r \le d \end{array} \right\}$ $dr, d\theta$

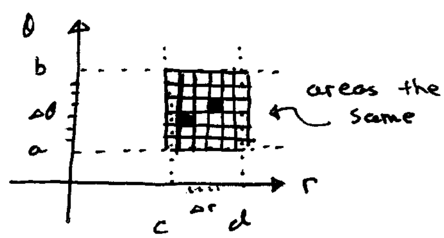
$$\iint_D f(x,y)\, dA = \int_a^b \left[\int_c^d f(\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})\, r\, dr \right] d\theta$$

extra term: r

Why extra term r?



area = $r\Delta r\Delta\theta$



Solution to previous problem:

$$\iint_D \arctan(y/x)\, dA \Rightarrow \int_0^{\pi/4} \left[\int_1^2 \arctan\left(\frac{r\sin\theta}{r\cos\theta}\right) r\, dr \right] d\theta$$

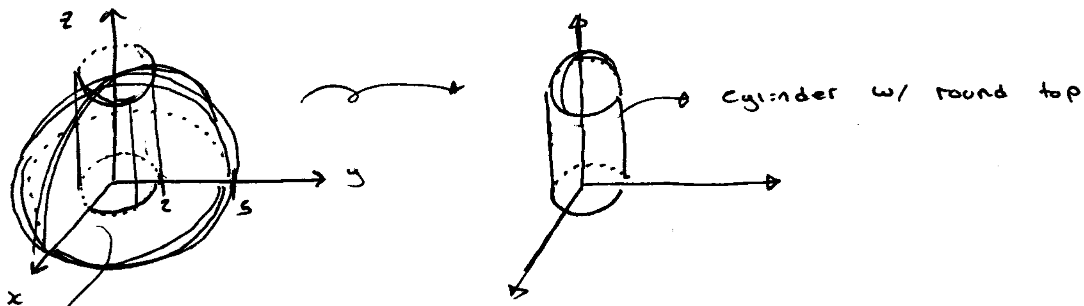
$$\Rightarrow \int_0^{\pi/4} \int_1^2 \cancel{\arctan}(\cancel{\tan}\,\theta)\, r\, dr\, d\theta$$

$$\Rightarrow \int_0^{\pi/4} \int_1^2 \theta r\, dr\, d\theta$$

$$\Rightarrow \int_0^{\pi/4} \left[\theta \left[\frac{r^2}{2} \Big|_{r=1}^{r=2} \right] \right] d\theta \rightsquigarrow (3/4)(\pi/16)^2 \text{ eventually..}$$

Example: Find the volume of the solid within the cylinder $x^2 + y^2 = 4$, below the hemisphere $z = \sqrt{25 - x^2 - y^2}$ and above $z = 0$

Solution: $\longrightarrow z^2 + x^2 + y^2 = 25$



$$\text{Volume} = \iint_D \sqrt{25 - x^2 - y^2}\, dA$$

disc of radius **2** in xy-plane $\Rightarrow \left\{ (r,\theta) \Rightarrow \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{matrix} \right\}$

$$\iint_D \sqrt{25-x^2-y^2}\, dA \longrightarrow \int_0^{2\pi}\int_0^2 \sqrt{25 - r^2} \cdot \underline{r}\, dr\, d\theta$$

where $u = r^2$

$du = 2r\, dr$

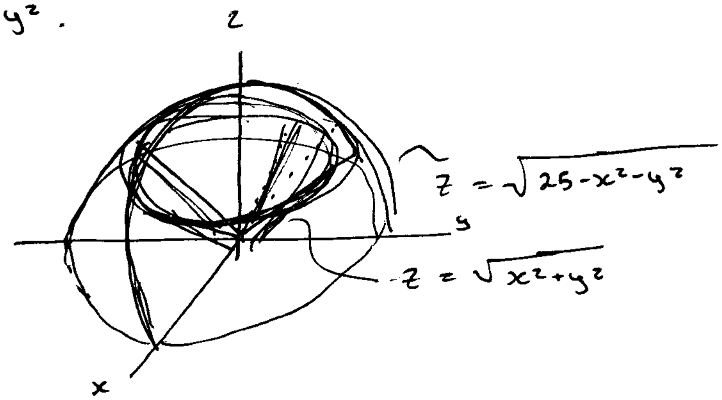
$dr = \frac{du}{2r}$

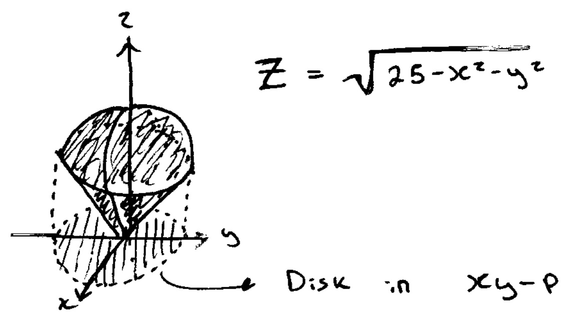
$$\Rightarrow \int_0^{2\pi}\int_{25}^{21} \sqrt{u} \cdot (-1/2)\, du\, d\theta \Rightarrow \int_0^{2\pi}\int_{21}^{25} \frac{1}{2}\sqrt{u}\, du\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} \Bigg|_{u=21}^{u=25} d\theta$$

Example: Find the volume of the solid below the hemisphere $z = \sqrt{25 - x^2 - y^2}$ and above the cone $z = \sqrt{x^2 + y^2}$.

Solution:





Disk in xy-plane of radius $r = 5/\sqrt{2}$

Intersection between

$$\left.\begin{aligned} z &= \sqrt{(25) - (x^2+y^2)} \quad \text{SPHERE} \\ z &= \sqrt{x^2+y^2} \quad \text{CONE} \end{aligned}\right\} \rightarrow \sqrt{25 - (x^2+y^2)} = \sqrt{x^2+y^2}$$

$$\hookrightarrow 25/2 = r$$

$$r = 5/\sqrt{2}$$

$$\text{Volume} = \iint_D \sqrt{25 - x^2 + y^2}\, dA - \iint_D \sqrt{x^2+y^2}\, dA$$

$$\Rightarrow \iint_D \left(\sqrt{25 - (x^2+y^2)} - \sqrt{x^2+y^2} \right) dA$$

$\hookrightarrow$ to polar coordinates