Oct. 2/18

Functions in Several Variables

Calculus I + II : $f : \mathbb{R} \to \mathbb{R}$

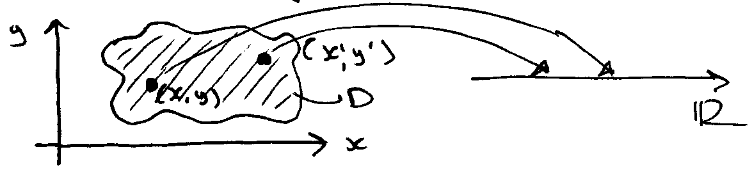
$f(x)$ = a formula in terms of x

$f(x, y)$ = a formula in terms of x, y

Now: $f : D = \text{domain in } \mathbb{R}^2 \longrightarrow \mathbb{R}$

INPUT: OUTPUT:

Domain D in $\mathbb{R}^2$



Ex: Location on globe = f(latitude, longitude)

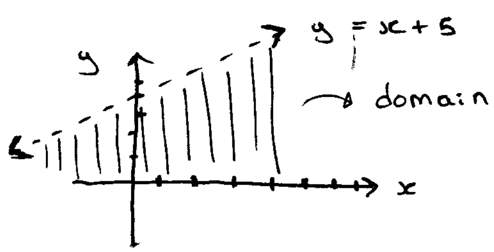
Ex: Sketch the domain (that is, all the points in $\mathbb{R}^2$ where the formula of the function make sense) for:

(1) $f(x, y) = \ln(x - y + 5)$

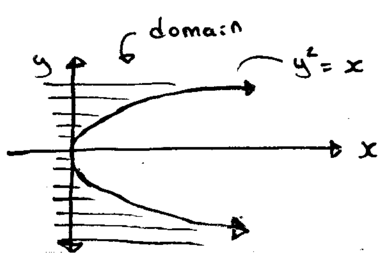
(2) $g(x, y) = \sqrt{y^2 - x}$

(3) $h(x, y) = \left(\dfrac{\sqrt{4 - 2x^2 - y^2}}{\ln x} \right)$

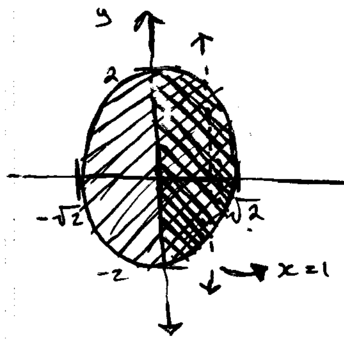
Solution : (1) domain of $f\{(x, y) : x - y + 5 > 0$
$x + 5 > y\}$



(2) domain of $g = \{(x, y) : y^2 - x \geq 0$
$y^2 = x\}$



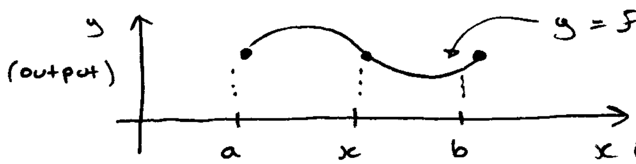
(3) domain of $h = \{ (x,y) : 4 - 2x^2 - y^2 \geq 0$ and $x > 0$ and $\ln x \neq 0 \therefore x \neq 1 \}$



$$\begin{cases} 4 - 2x^2 - y^2 = 0 \\ 4 - 2x^2 - y^2 \geq 0 \\ x > 0 \\ x \neq 1 \end{cases}$$

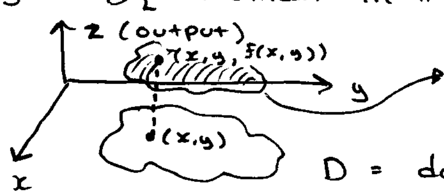
Graph of a Function in 2 variables

Calculus I and II : $f : [a, b] \to \mathbb{R}$



$y = f(x)$ = a curve in 2-dim (graph of f)

Now: $f : D[= \text{Domain in } \mathbb{R}^2] \longrightarrow \mathbb{R}$



Graph : $z = f(x,y)$
a surface in 3-dim.

D = domain in x-y plane (INPUT)

Graph of f =

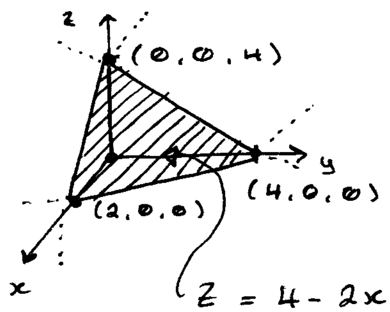
$$= \{(\underbrace{x, y}_{\text{INPUT}}, \underbrace{z}_{\text{OUTPUT}}) : z = f(x,y) \text{ and } (x,y) \text{ in } D\}$$

Ex: Sketch the graph of :

(1) $f(x,y) = 4 - 2x - y$

(2) $g(x,y) = 4 - x^2 - y^2$

Solution: (1) $f(x,y) = 4 - 2x - y$

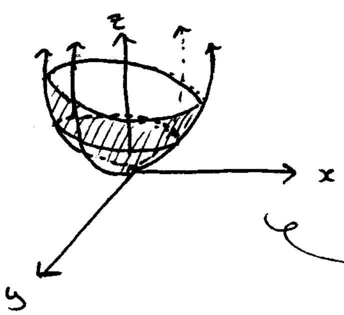


$Z = 4 - 2x - y$

Graph of $f = \{(x,y,z) : z = \underbrace{4 - 2x - y}_{f(x,y)}\}$

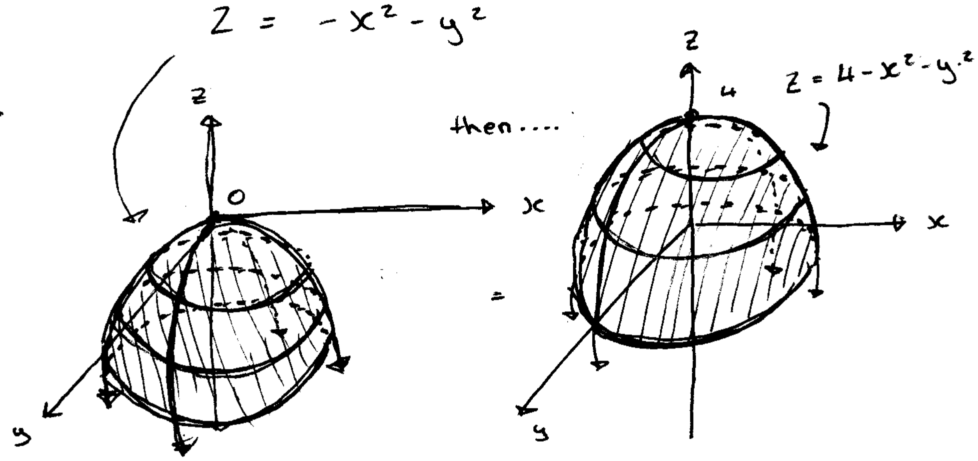
$2x + y + z = 4$ (PLANE)

(2) $g(x,y) = 4 - x^2 - y^2$



Graph of $g = \{(x,y,z) : z = \underbrace{4 - x^2 - y^2}_{f(x,y)}\}$

$Z = -x^2 - y^2$



$z = 4 - x^2 - y^2$

How do we figure out the graph of:

(1) $f(x,y) = x - y^2$

(2) $g(x,y) = e^{y/x}$

Remark: For the harder examples we can use the concept of a "level curve" (From maps!)

K - level curve = $\{(x,y) : f(x,y) = K\}$

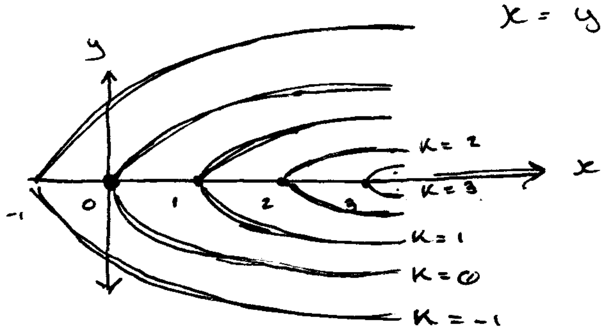
Favorite number

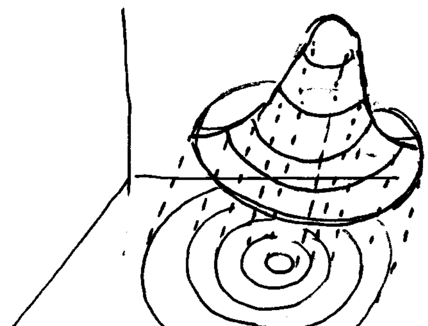
Sol : $f(x,y) = x - y^2$

K - level curve $= \{ (x,y) : f(x,y) = k$

$x - y^2 = k$

$x = y^2 + k \}$





(2) $g(x,y) = e^{y/x}$

k-level curves =

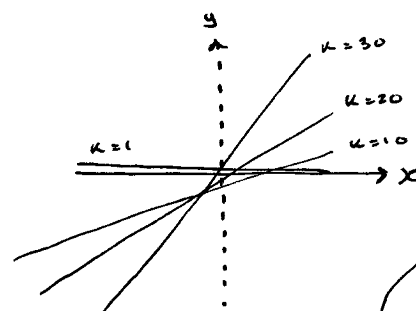
$= \{ (x,y) : e^{y/x} = k$

$y/x = \ln k$

$\boxed{y = (\ln k) x}$

line with slope $\ln k$

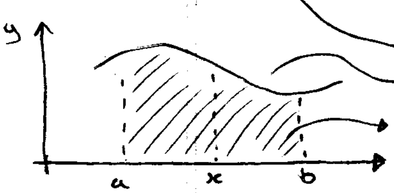
((x,y) with $x = 0$ are not in the domain)



At end: (Reminder)

Calculus II

$f : [a,b] \to \mathbb{R}$



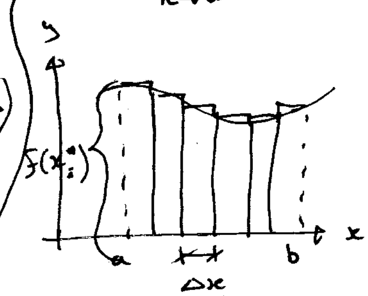
→ $y = f(x)$ Graph of f

→ Area under $y = f(x)$ $= \int_a^b f(x)\,dx$

also at end:

$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$

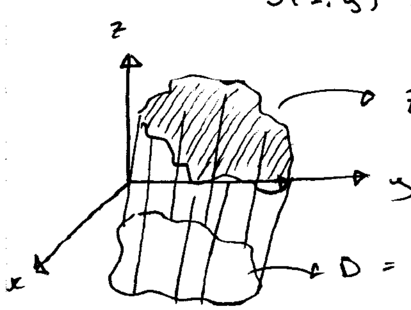
heights, length



Multiple Integration (= integration of functions in several variables)

GOAL : Let $f : D[\text{Domain in } \mathbb{R}^2] \longrightarrow \mathbb{R}$ be a function in 2 variables, with

$f(x,y) \geq 0$ for (x,y) in D



→ $z = f(x,y)$ graph of f

→ D = domain in x-y plane

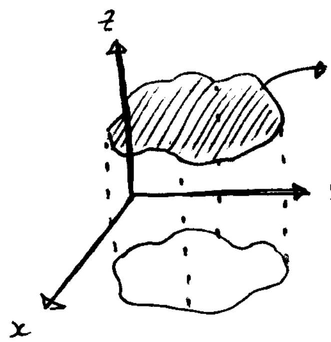
We are interested in computing the volume of the solid below the graph $z = f(x,y)$ and above x-y plane

Double integrals (= integration of functions in 2 variables)

GOAL: Given $f: D \longrightarrow \mathbb{R}$
"domain in $\mathbb{R}^2$"

with $f(x,y) \geq 0$ for all (x,y) in D

we want to compute the volume of the solid



$z = f(x,y)$
Graph of f

D = domain in x-y plane

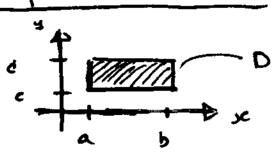
$$V = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i$$

Mathematical definition of Double integral

$$\text{"} \iint_D f(x,y)\,dA \text{"} \stackrel{\text{def.}}{=} \lim_{i=1} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i$$

Computation of Double Integrals

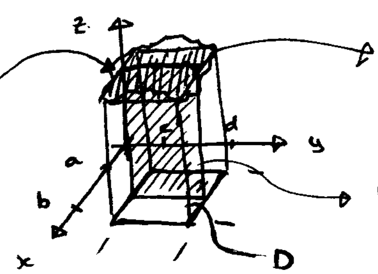
Special case: D = domain in x-y plane is a rectangle



$$\Rightarrow \left\{ (x,y) : \begin{matrix} a \leq x \leq b \\ c \leq y \leq d \end{matrix} \right\}$$

$f: D \longrightarrow \mathbb{R}$
"rectangle in x-y plane"
$f(x,y) \geq 0$, (x,y) in D

Curve of intersection
$z = f(x,y)$
Fixed



$z = f(x,y)$
(whole surface)

$A(x)$: area of cross section

From Calculus II :

$$\iint_D f(x,y)\,dA = \text{volume}$$

$$\rightarrow \int_a^b A(x)\,dx = \int_a^b \int_c^d [f(x,y)\,dy]\,dx \quad \text{integrated integrals}$$

we can switch the roles of x and y in this arrangement:

$$\iint_D f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,d\underset{\text{Fixed}}{x}\right] dy$$

→ **Ex:** Compute $\iint_D (x/y + y/x)\,dA$ where $D = \left\{ (x,y) : \begin{array}{l} 1 \le x \le 4 \\ 1 \le y \le 2 \end{array} \right\}$

Solution : $\iint_D \left(\frac{x}{y} + \frac{y}{x}\right) dA = \int_1^4 \left[\int_1^2 \frac{\overset{\text{Fixed}}{x}}{y} + \frac{y}{\underset{\text{Fixed}}{x}}\,dy \right] dx$

First : $\int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy = \left[x \ln y + 1/x \cdot y^2/2 \right]_{y=1}^{y=2}$

$\Rightarrow \left[x \ln 2 + \frac{1}{x} \cdot 2 \right] - \left[\cancel{x \ln 1} + \frac{1}{x} \cdot \frac{1}{2} \right]$

$\Rightarrow x \ln 2 + (3/2)(1/x)$

$$\iint_D (x/y + y/x)\,dA = \int_1^4 (x \ln 2 + (3/2)(1/x))\,dx = \left[\frac{x^2}{2} \cdot \ln 2 + 3/2 \ln x \right]_{x=1}^{x=4}$$

$\Rightarrow 8 \ln 2 + (3/2) \ln 4 - (1/2) \ln 2$

→ **Ex:** Compute $\iint_D y \sin(xy)\,dA$ where $D = \left\{ (x,y) : \begin{array}{l} 1 \le x \le 2 \\ 0 \le y \le \pi \end{array} \right\}$

Solution 1 : $\iint_D y \sin(xy)\,dA = \int_0^\pi \left[\int_1^2 y \sin(xy)\,dx \right] dy$

First : $\int_1^2 y \sin(xy) = \left[\cancel{y} \cdot \left(\frac{-\cos(xy)}{\cancel{y}} \right) \right]_{x=1}^{x=2}$

$\Rightarrow -\cos(xy) \Big|_{x=1}^{x=2} \Rightarrow -\cos(2y) - (-\cos(y))$

$\Rightarrow \cos(y) - \cos(2y)$

Finally,

$$\iint_D y \sin(xy)\,dA = \int_0^\pi (\cos y - \cos(2y))\,dy = \sin y - \frac{\sin 2y}{2} \Bigg|_{y=0}^{y=\pi}$$

$\Rightarrow \sin \pi - \frac{\sin(2\pi)}{2} = \boxed{0}$

Solution 2: $\iint_D y\sin(xy)\,dA = \int_1^2 \left[\int_0^\pi y\sin(\underset{\text{Fixed}}{x}y)\,dy\right]dx$

where $\int u\,dv = u\cdot v - \int v\,du$

→ First: $\int_0^\pi \underset{u}{y}\, \underset{dv}{\sin(xy)\,dy} = \underset{u}{y}\left(\underset{v}{\frac{-\cos(xy)}{x}}\right)\Bigg|_{y=0}^{y=\pi} - \int_0^\pi \underset{v}{-\frac{\cos(xy)}{x}}\,\underset{du}{dy}$

$\Rightarrow (-\pi/x)\cos(\pi x) + \int_0^\pi \frac{\cos(xy)}{x}\,dy$

$\Rightarrow -\pi/x\cos(\pi x) + 1/x \cdot \frac{\sin(xy)}{x}\Big|_{y=0}^{y=\pi}$

$\Rightarrow -\pi/x\cos(\pi x) + \sin(\pi x)/x^2$

→ Second: $\iint_D y\sin(xy)\,dA = \int_1^2 -\pi/x\cos(\pi x) + \frac{\sin(\pi x)}{x^2}\,)\,dx$

$$\Rightarrow \underbrace{\int_1^2 -\pi/x\cos(\pi x)\,dx}_{1} + \underbrace{\int_1^2 \frac{\sin(\pi x)}{x^2}\,dx}_{2}$$

(1) $= -\pi\int_1^2 \frac{1}{x}\cos(\pi x)\,dx \Rightarrow -\pi\left[\frac{1}{x}\cdot\frac{\sin(\pi x)}{\pi}\right]_{x=1}^{x=2} - \int_1^2 \frac{\sin(\pi x)}{\pi}\left(\frac{-1}{x^2}\right)dx$

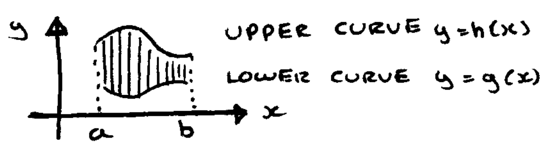
$u = 1/x \qquad v = \int\cos(\pi x)\,dx = \frac{\sin(\pi x)}{\pi}$

$du = -1/x^2 \qquad dv = \cos(\pi x)\,dx$

$\Rightarrow -\cancel{\pi}\int_1^2 \frac{\sin(\pi x)}{\cancel{\pi}x^2}\,dx = -\int_1^2 \frac{\sin(\pi x)}{x^2}\,dx = -(2)$

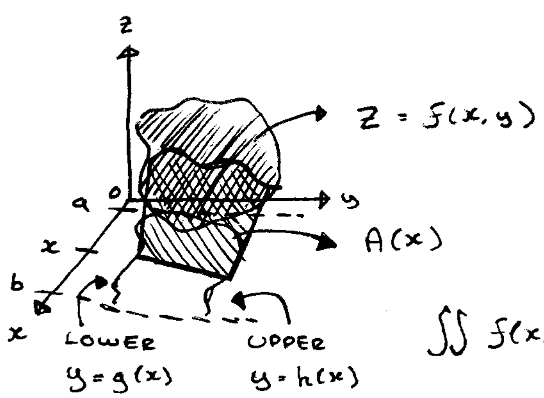
Double integrals over more general domains

(1) D = type I domain = $\left\{ (x,y) \;\middle|\; \begin{array}{l} a \le x \le b \\ g(x) \le y \le h(x) \end{array} \right\}$



Given $f: D \longrightarrow R$, $f(x,y) \ge 0$ for (x,y) in D

how to compute $\iint_D f(x,y)\,dA$?



$\int_{g(x)}^{h(x)} f(\underset{\text{Fixed}}{x},y)\,dy$

$\iint f(x,y)\,dA = \int_a^b A(x)\,dx \int_a^b \left[\int_{g(x)}^{f(x)} f(x,y)\,dy\right]dx$

#7 $\rightarrow r = 2\cos\theta \,|\, r$

$r^2 = 2r\cos\theta$

$x^2 + y^2 = 2x$

$x^2 - 2x + y^2 = 0$ $\rightarrow$

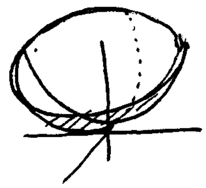
$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$

$\rightarrow$ Shifted circular cylinder

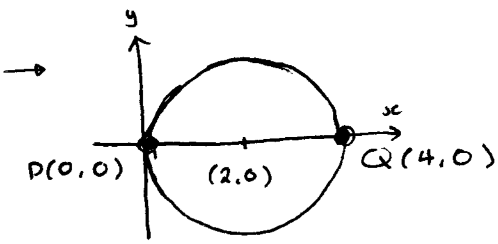
$\rightarrow$ $z = r^2$

$z = x^2 + y^2$



Paraboloid

#6 $\rightarrow$



$\rightarrow$ $(x^2+2) + y^2 = 4$

Parameterization:

$x = f(t)$

$y = g(t)$

$$\begin{cases} x - 2 = 2\cos\theta \\ y = 2\sin\theta \end{cases} \longrightarrow \begin{cases} x = 2 + 2\cos\theta \\ y = 2\sin\theta \end{cases}$$

$0 \le \theta \le 2\pi$

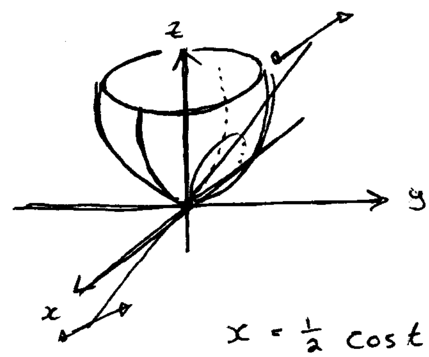
counterclockwise

Ex. Parameterize the curve of intersection between $\underbrace{z = x^2+y^2}_{\text{Paraboloid}}$ and $\underbrace{z = y}_{\text{Plane}}$

$x = f(t)$

$y = g(t)$

$z = h(t)$



Solution #1:

$$\left.\begin{array}{l} z = x^2+y^2 \\ z = y \end{array}\right\} \quad \begin{array}{l} y = x^2+y^2 \\ 0 = x^2+y^2-y \end{array}$$

$[y^2 - 2y(1/2) + (1/2)^2]$

$1/4 = x^2 + y^2 - y + 1/4$

$1/4 = x^2 + (y - 1/2)^2$

$x = \frac{1}{2}\cos t$

$y - 1/2 = 1/2 \sin t$

$$\begin{cases} x = 1/2 \cos t \\ y = 1/2 + 1/2 \sin t \\ z = 1/2 + 1/2 \sin t \end{cases}$$

$0 \le t = 2\pi$

Solution #2 : Use cylindrical coordinates

$$C: \begin{cases} z = x^2 + y^2 \\ z = y \end{cases} \rightsquigarrow \begin{cases} z = r^2 \\ z = r\sin\theta \end{cases}$$

$r^2 = r\sin\theta \longrightarrow r = \sin\theta$

Cylindrical coordinates in general

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

For C: I have constraint $r = \sin\theta$

$$x = \sin\theta\cos\theta = \sin\theta\cos\theta$$
$$y = \sin\theta\sin\theta = \sin^2\theta$$
$$z = \sin^2\theta \qquad 0 \le \theta \le 2\pi$$