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Vector Valued Functions

Goal: To study more general curves in $\mathbb{R}^3$

Remember: Calculus I + II $\quad f: \mathbb{R} \to \mathbb{R}$

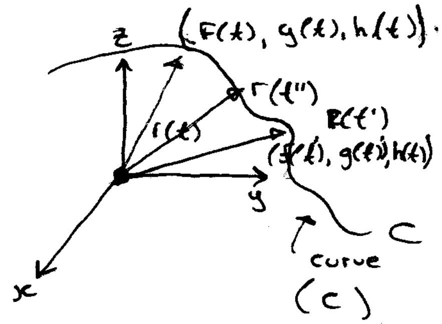
$f(x)$ = expression in terms of x

Now: $r(t)$

input: real number

$\Rightarrow \langle f(t), g(t), h(t) \rangle$

output: a vector in $\mathbb{R}^3$



Def'n: $r(t)$ is called a vector valued function

Remark: $r(t) = \langle f(t), g(t), h(t) \rangle$

then
$$\left.\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ g &: \mathbb{R} \to \mathbb{R} \\ h &: \mathbb{R} \to \mathbb{R} \end{aligned}\right\} \text{ component functions}$$

We can think about $r(t)$ = the position vector that follows the path of a particle in $\mathbb{R}^3$, at time t

Vector value Function $r(t) = \langle f(t), g(t), h(t) \rangle \longleftrightarrow$ curve C in $\mathbb{R}^3$, which is traced out by the tip of the vectors $r(t)$

PARAMETRIC EQUATION OF C $\rightarrow$
$$\boxed{\begin{aligned} x &= f(t) \\ y &= g(t) \\ z &= h(t) \end{aligned}} \quad t = \text{parameter}$$

Ex: $r(t) = \langle \underbrace{\ln(t-1)}_{f(t)}, \underbrace{\sqrt{5-t}}_{g(t)}, \underbrace{t^2}_{h(t)} \rangle$

Find the domain of $r(t)$.

Solution: we have to find all t such that $f(t), g(t), h(t)$ make sense.

Constraints: $\left\{\begin{aligned} &t-1 > 0 \\ &\text{and} \\ &5-t \geq 0 \end{aligned}\right. \rightarrow \left\{\begin{aligned} &t > 1 \\ &\text{and} \\ &t \leq 5 \end{aligned}\right.$

Domain: $(1, 5]$

Ex:

Sketch the curves associated to the following vector valued functions:

(1) $r(t) = \langle 2-t, 4-3t, -1+t \rangle$

(2) $r(t) = \langle \cos t, \sin t, t \rangle = \cos t\, i + \sin t\, j + t\, k$

(3) $r(t) = \langle t^2, t^4, t^6 \rangle = t^2 i + t^4 j + t^6 k$

Solution: (1) we need to find C

$x = 2-t$

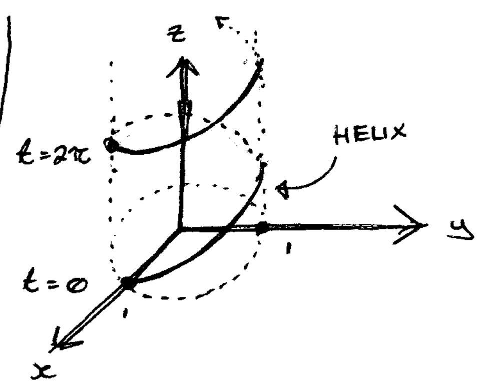
$y = 4-3t$ $\quad t = \text{parameter}$ $\quad$ Domain: $(-\infty, \infty)$

$z = -1+t$

C = line containing point $(2, 4, -1)$
with direction vector $\langle -1, -3, 1 \rangle$

(2) $r(t) = \langle \cos t, \sin t, t \rangle = \cos t\, i + \sin t\, j + t\, k$

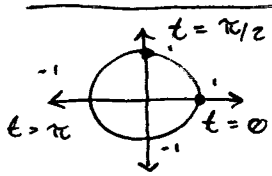
we are looking at C $\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$



in 2-dimension

$x = \cos t$

$y = \sin t$



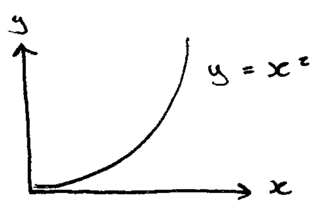
(3) $r(t) = \langle t^2, t^4, t^6 \rangle$

C: $\begin{cases} x = t^2 \\ y = t^4 \\ z = t^6 \end{cases}$ $\quad t = \text{parameter}$

in 2-dimension (x-y):

$x = t^2$

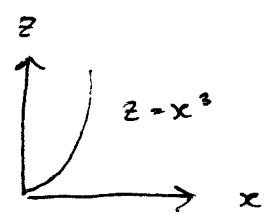
$y = t^4 = x^2$



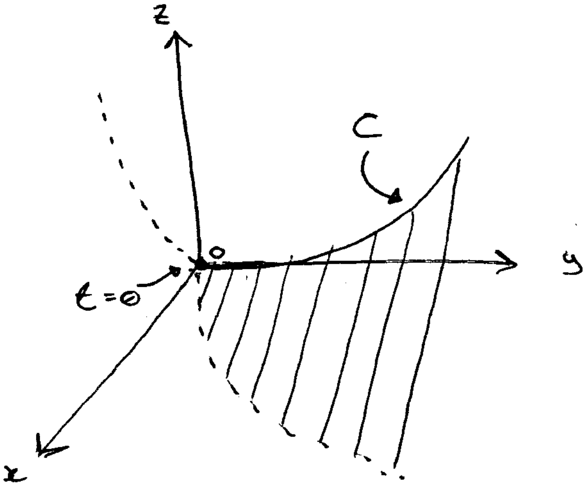
in 2-dimension (x-z):

$x = t^2$

$z = t^6 = x^3$



Now in $\mathbb{R}^3$



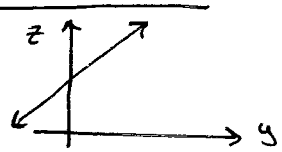
Ex:

Find a vector valued function associated to the curve C of intersection between $z = \sqrt{x^2+y^2}$ and $z = 1+y$

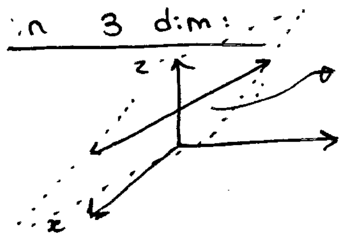
Solution:

$z = 1+y$

in 2 dim:



in 3 dim:

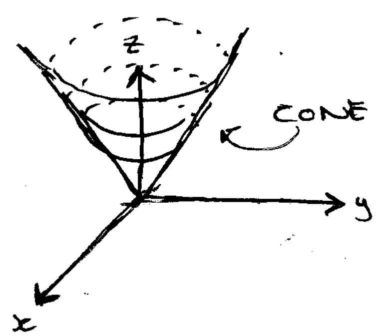


plane (x unconstrained)

$z = \sqrt{x^2+y^2}$

- intersection with y-z-plane $[x = 0]$

$\begin{cases} z = \sqrt{x^2+y^2} \\ x = 0 \end{cases} \rightsquigarrow z = \sqrt{y^2}$
$z = y$
$z = -y$

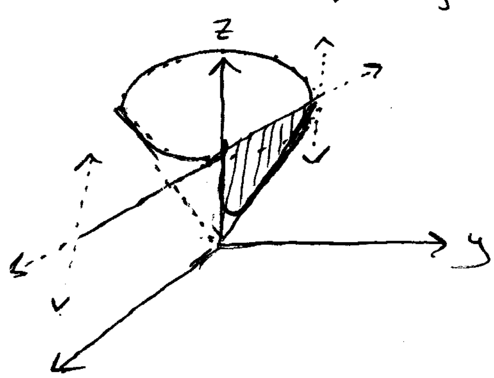


- intersection with x-z plane $[y = 0]$

$\begin{cases} z = \sqrt{x^2+y^2} \\ y = 0 \end{cases} \rightsquigarrow z = \sqrt{x^2}$
$z = x$
$z = -x$

- intersection with $z = k$ ← fav. number

$\begin{cases} z = \sqrt{x^2+y^2} \\ z = k \end{cases} \rightsquigarrow k = \sqrt{x^2+y^2}$ ← circle of radius k



(or something similar)

actually

Solution: $C: \begin{cases} z = \sqrt{x^2+y^2} & \text{(cone)} \\ z = 1+y & \text{(plane)} \end{cases}$

$\sim \quad \sqrt{x^2+y^2} = 1+y$

$x^2+y^2 = (1+y)^2 \quad \to \quad x^2 + \cancel{y^2} = 1 + 2y + \cancel{y^2}$

$x^2 = 1 + 2y$

$y = \frac{1}{2}(x^2 - 1)$

parabola

$C: \quad x = f(t)$

$y = g(t)$

$z = h(t)$

Take for example $\boxed{x = t}$

Then $y = \frac{1}{2}(x^2-1) = \frac{1}{2}(t^2-1)$

$z = 1+y = 1 + \frac{1}{2}(t^2-1) = \frac{1}{2} + \frac{1}{2}t^2$

Answer:

$$\boxed{\begin{aligned} x &= t \\ y &= \tfrac{1}{2}(t^2-1) \\ z &= \tfrac{1}{2} + \tfrac{1}{2}t^2 \end{aligned}}$$

For C

OR

$\boxed{r(t) = \langle t, \frac{1}{2}(t^2-1), \frac{1}{2} + \frac{1}{2}t^2 \rangle}$

vector valued Function

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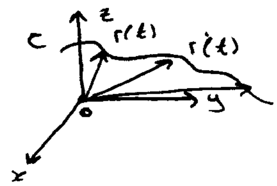
Vector values Functions

$r(t) = \langle f(t), g(t), h(t) \rangle \longleftrightarrow$ A curve C in $\mathbb{R}^3$

scalar



Parametric eq'n of C:

$$\boxed{\begin{aligned} x &= f(t) \\ y &= g(t) \\ z &= h(t) \end{aligned}}$$

t = parameter

Remark: For $r(t) = \langle f(t), g(t), h(t) \rangle$

if $f: \mathbb{R} \to \mathbb{R}$

$g: \mathbb{R} \to \mathbb{R}$ are continuous then

$h: \mathbb{R} \to \mathbb{R}$

the curve C is continuous (no jumps!)

Moreover,

$$\lim_{t\to t_0} r(t) = \langle \lim_{t\to t_0} f(t), \lim_{t\to t_0} g(t), \lim_{t\to t_0} h(t) \rangle$$

Derivatives of Vector Valued Functions

Def. For $r(t) = \langle f(t), g(t), h(t) \rangle = f(t)i + g(t)j + h(t)k$

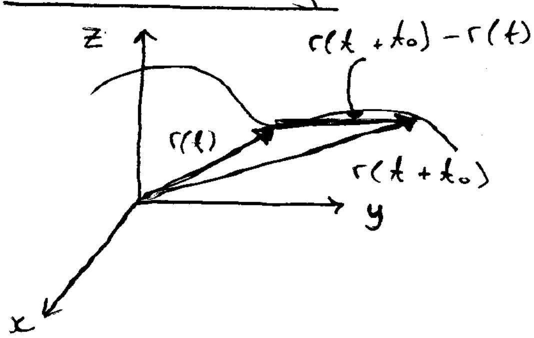
we define $r'(t) = \lim_{t_0 \to 0} \frac{1}{t_0} \left[r(t+t_0) - r(t) \right]$

scalar   vector

if the limit exists,

$r'(t)$ = another vector valued function

Geometrically:



$r'(t)$ gives the direction of the tangent line to C

<u>Computationally</u>: $r(t) = \langle f(t), g(t), h(t) \rangle$

If $f, g, h : \mathbb{R} \to \mathbb{R}$ are differentiable at $\underline{\underline{t}}$ then,

$r'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Indeed, $r'(t) = \lim_{t_0 \to 0} \frac{1}{t_0}\left[r(t+t_0) - r(t)\right]$

$= \lim_{t \to t_0} \frac{1}{t_0}\left[\langle f(t+t_0), g(t+t_0), h(t+t_0)\rangle - \langle f(t), g(t), h(t)\rangle\right]$

$= \lim_{t \to t_0} \langle \frac{1}{t_0}(f(t+t_0) - f(t)), \frac{1}{t_0}(g(t+t_0) - g(t)), \frac{1}{t_0}(h(t+t_0) - h(t)) \rangle$

$= \langle f'(t), g'(t), h'(t) \rangle$

<u>Ex</u>: Find the equation of the tangent line to the curve C associated to

$$r(t) = \cos t \cdot i + \sin t \cdot j + \ln(\cos t) \cdot k$$

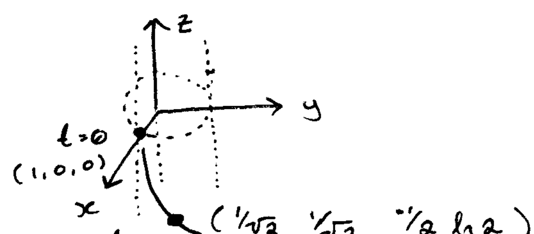
at the point $(1/\sqrt{2}, 1/\sqrt{2}, -\frac{1}{2}\ln 2)$ on C

<u>Solution</u>: $r(t) = \cos t\, i + \sin t\, j + \ln(\cos t)\, k$

C: $x = \cos t$

$y = \sin t$

$z = \ln(\cos t)$



$t \in [0, \pi/2)$ part of domain of t

First, notice that the point $(1/\sqrt{2}, 1/\sqrt{2}, -\frac{1}{2}\ln 2)$ corresponds to $t = ?$

$$\left.\begin{array}{l} 1/\sqrt{2} = x = \cos t \\ 1/\sqrt{2} = y = \sin t \end{array}\right\} t = \pi/4$$

$-\frac{1}{2}\ln 2 = z = \ln(\cos t)$

$\ln 1/\sqrt{2} = \ln (1/2)^{1/2}$

$\Rightarrow \frac{1}{2}\ln(1/2)$

$= -\frac{1}{2}\ln 2$

Second, the direction of tangent line is given by

$r'(t) = -\sin t\, i + \cos t \cdot j + \frac{1}{\cos t}\cdot(-\sin t)\cdot k$

$r'(t) = -\sin t \cdot i + \cos t \cdot j - \tan t \cdot k$

Tangent line to C at $t = \pi/4$

$r'(\pi/4) = -1/\sqrt{2}\, i + 1/\sqrt{2}\, j - 1k = \langle -1/\sqrt{2}, 1/\sqrt{2}, -1 \rangle$

(Final answer, from prev:)

Equation of tangent line to C at $\left(\overset{x_0}{1/\sqrt{2}}, \overset{y_0}{1/\sqrt{2}}, \overset{z_0}{-1/2 \ln 2} \right)$

$$\begin{cases} x = 1/\sqrt{2} - 1/\sqrt{2}\, t \\ y = 1/\sqrt{2} + 1/\sqrt{2}\, t \\ z = -1/2 \ln 2 - t \end{cases}$$

Def'n: $r(t) = \langle f(t), g(t), h(t) \rangle$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \text{tangent unit vector}$$

perpendicular(?)

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \text{normal unit vector}$$

Remark: (i) $T(t)$: has direction of $r'(t)$

tangential vector

normal vector $N(t)$: has direction of $T'(t)$

Question: Does that mean that $N(t)$ has the direction of $r''(t)$?

Answer: No - in general!

Ex: $r(t) = 1i + tj + t^2 k$

$r'(t) = 1j + 2tk$

$\boxed{r''(t) =}$ $\boxed{2k}$ has direction of vector k

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{1}{\sqrt{1+4t^2}} (1j + 2t \cdot k)$$

$$T(t) = \frac{1}{\sqrt{1+4t^2}} \cdot j + \frac{2t}{\sqrt{1+4t^2}} \cdot k$$

$N(t)$ has the same direction as $T'(t)$

$T'(t) = -1/2 (1+4t^2)^{-3/2} \cdot 8t\, j + (\cdots) k$ ← both j and k components

Properties of derivative of vector valued functions

u, v = vector valued functions

$$\frac{d}{dt} \underbrace{u(t)v(t)}_{\text{scalar}} = \underbrace{u'(t) \cdot v(t)}_{\text{scalar}} + u(t)v'(t)$$

$$\big[\underbrace{u(t) \times v(t)}_{\text{vector}}\big]' = u'(t) \times v(t) + u(t) \times v'(t)$$

Remark: $N(t)$ is perpendicular to $T(t)$:

$\|T(t)\| = 1 \quad \therefore \quad \|T(t)\|^2 = 1$

$T(t) \cdot T(t) = 1$

By taking derivatives on both sides:

$T'(t) \cdot T(t) + T(t) \cdot T'(t) = 0$

$\cancel{2}\, T'(t) \cdot T(t) = 0$

same direction

$T'(t)$ is perpendicular to $T(t)$

$N(t)$

Ex: Compute $N(t)$ for:

$r(t) = \cos(t) \cdot i + \sin(t) \cdot j + \ln(\cos t) \cdot k$

Sol'n: $T(t) = \dfrac{r'(t)}{\|r'(t)\|}$

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

$r'(t) = -\sin t \cdot i + \cos t\, j + \frac{1}{\cos t} \cdot (-\sin t) \cdot k$

$= -\sin t\, i + \cos t\, j - \tan t \cdot k$

$\|r'(t)\| = \sqrt{(\sin t)^2 + (\cos t)^2 + (\tan t)^2}$

$\Rightarrow \sqrt{1 + (\tan t)^2} \Rightarrow \dfrac{1}{\cos t}$

$T(t) = \dfrac{1}{\|r'(t)\|} r'(t) = \cos t \left[-\sin t\, i + \cos t \cdot j - \tan t\, k \right]$

$\Rightarrow -\sin t \cos t \cdot i + \cos^2 t \cdot j - \sin t \cdot k$

$T(t) = -\dfrac{\sin(2t)}{2} i + \cos^2 t\, j - \sin t \cdot k$

$T'(t) = -\cos(2t) \cdot i + 2\cos t \cdot (-\sin t)\, j - \cos t\, k$

$T'(t) = -\cos(2t)\, i - \sin(2t)\, j - \cos t\, k$

$\|T'(t)\| = \sqrt{[\cos(2t)]^2 + [\sin(2t)]^2 + \cos^2 t}$

$= \sqrt{1 + \cos^2 t}$

$N(t) = \dfrac{T'(t)}{\|T'(t)\|}$

$= -\dfrac{\cos(2t)}{\sqrt{1+\cos^2 t}} \cdot i \ldots$

$\ldots - \dfrac{\sin(2t)}{\sqrt{1+\cos^2 t}} \cdot j \ldots$

$\ldots - \dfrac{\cos t}{\sqrt{1+\cos^2 t}} \cdot k$

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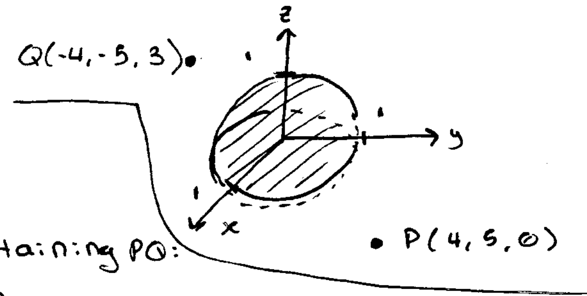
Q Is the point $(-4,-5,3)$ visible from the point $(4,5,0)$ if there is an opaque ball of radius 1 centered at the origin?

$P(4,5,0)$

$Q(-4,-5,3)$



Sol'n:

Eq'n of the line containing PQ:

- point $P(4,5,0)$
- direction $v = \overrightarrow{QP} = \langle 4-(-4),\ 5-(-5),\ 0-3 \rangle = \langle 8, 10, -3 \rangle$

$$\boxed{\begin{aligned} x &= 4 + 8t \\ y &= 5 + 10t \\ z &= 0 - 3t \end{aligned}}$$

Eq'n of the sphere:

$$x^2 + y^2 + z^2 = 1$$

For intersection

$$\underbrace{(4+8t)^2}_{x} + \underbrace{(5+10t)^2}_{y} + \underbrace{(-3t)^2}_{z} = 1$$

$$16 + 64t + 64t^2 + 25 + 100t + 100t^2 + 9t^2 = 1$$

$$173t^2 + 164t + 40 = 0$$

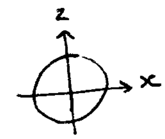
where $a = 173,\ b = 164,\ c = 40$

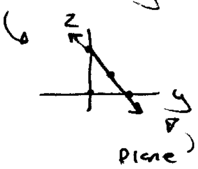
$$\Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \frac{-164 \pm \sqrt{(164)^2 - 4(173)(40)}}{2(173)}$$

negative, so no real roots

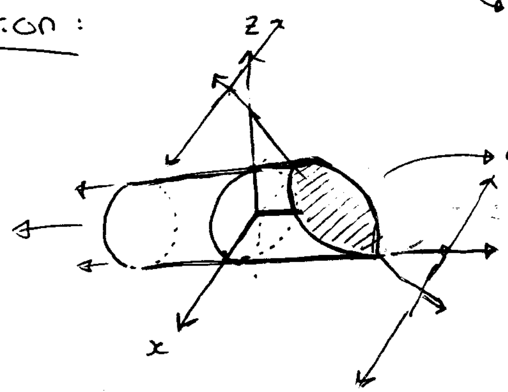
$\Rightarrow$ no real roots, no point of intersection with the sphere.

Q Find a parametric equation for the curve of intersection between the cylinder $x^2+z^2=1$ and plane $z=2-y$





Solution :



$x = f(t)$
$y = g(t)$
$z = h(t)$

Solution 1 :

Take $y = t$

Then $\left[z = 2 - y = 2 - t\right]$

$x^2 + z^2 = 1$

$x^2 + (2-t)^2 = 1$

$x^2 = 1 - (2-t)^2$

$x = \pm\sqrt{1-(2-t)^2}$

$x^2 + z^2 = 1$

And

$z = 2 - y$

$C = C_1$ and C_2

C_1	C_2
$x = \sqrt{1-(2-t)^2}$	$x = -\sqrt{1-(2-t)^2}$
$y = t$	$y = t$
$z = 2-t$	$z = 2-t$

DOMAIN $\{t : (2-t)^2 \le 1\}$

$-1 \le 2 - t \le 1$

$1 \le t \le 3$

Solution #2 :

$$\left[\begin{array}{l} x = \cos t \\ z = \sin t \end{array} \Big\rangle \; x^2 + z^2 = 1 \right.$$

$$\left. y = 2 - z = 2 - \sin t \right]$$

$t \in [0, 2\pi]$