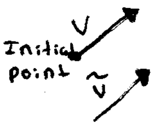
Sept. 11 / 18

Vectors in 3-dimensions

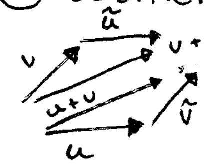
vector : mathematical object defined by < direction, magnitude



Remark : Two vectors with same direction and magnitude will be identical

Operations with vectors :

(1) addition



$v + u$ = another vector with

initial point = initial point of v

terminal point = terminal point of $\tilde{u}$

Parallelogramic Law: $u + v = v + u$

(2) multiplication by scalar

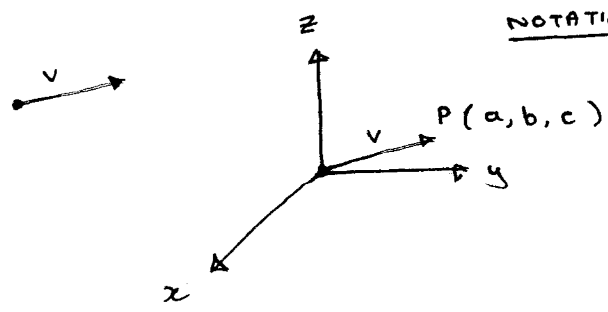
v = vector

λ = scalar

NOTATION : magnitude

λv = a vector with < magnitude $\|\lambda v\| = |\lambda| \|v\|$, direction < $\lambda > 0$: same, $\lambda < 0$: opposite

Components of vectors in $\mathbb{R}^3$



NOTATION: $v = \langle a, b, c \rangle$

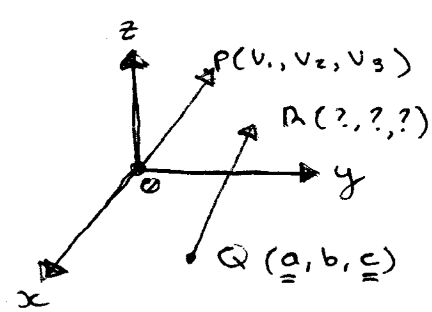
components of v

REMARK : $P(a, b, c)$ = a point in $\mathbb{R}^3$

$\langle a, b, c \rangle$ = a vector that starts at 0 and ends at $P(a, b, c)$

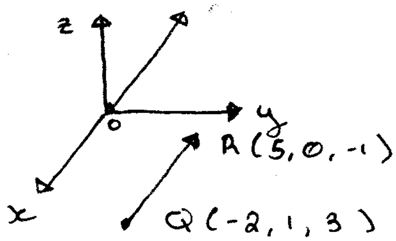
Question : Let $v = \langle v_1, v_2, v_3 \rangle$

Given a point $A(a, b, c)$, what are the coordinates of $R(?, ?, ?)$ such that $\overrightarrow{QR} = v$?



Answer: $R(v_1 + a, v_2 + b, v_3 + c)$

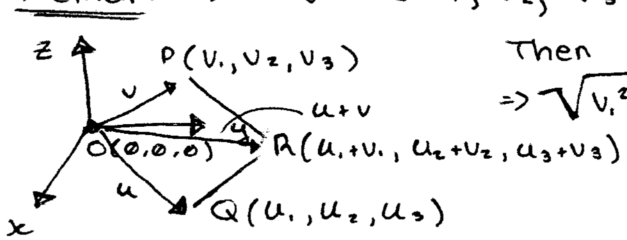
Example: Write the components of the vector $\overrightarrow{QR}$ where $Q(-2,1,3)$ and $R(5,0,-1)$.



Solution:

$$V = \overrightarrow{QR} = \langle 5-(-2),\ 0-1,\ -1-3 \rangle$$
$$= \langle 7, -1, -4 \rangle$$

Remark: $V = \langle V_1, V_2, V_3 \rangle$



Then $\|V\| = |OP| = \sqrt{(v_1-0)^2+(v_2-0)^2+(v_3-0)^2}$

$\Rightarrow \sqrt{V_1^2+V_2^2+V_3^2}$

Remark: Addition of two vectors: $u = \langle u_1, u_2, u_3 \rangle$

$V = \langle V_1, V_2, V_3 \rangle$

Then $u + v = \langle u_1+v_1,\ u_2+v_2,\ u_3+v_3 \rangle$

Similarly:

$\lambda V = \langle \lambda v_1, \lambda v_2, \lambda v_3 \rangle$

Definition $i = \langle 1,0,0 \rangle$

$j = \langle 0,1,0 \rangle$

$k = \langle 0,0,1 \rangle$

Notice:

$V = \langle V_1, V_2, V_3 \rangle$

$= \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle \ldots$

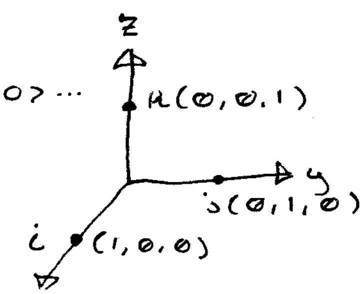
$\ldots + \langle 0, 0, v_3 \rangle$

$= V_1 \langle 1, 0, 0 \rangle \ldots$

$\ldots + V_2 \langle 0, 1, 0 \rangle \ldots$

$\ldots + V_3 \langle 0, 0, 1 \rangle \ldots$

$= V_1 i + V_2 j + V_3 k$



Dot Product between 2 vectors

$u = \langle u_1, u_2, u_3 \rangle = u_1 i + u_2 j + u_3 k$

$V = \langle V_1, V_2, V_3 \rangle = V_1 i + V_2 j + V_3 k$

DEF: $\boxed{u \cdot V}$ = a scalar

$= u_1 V_1 + u_2 V_2 + u_3 V_3$

Properties (1) $u \cdot v = v \cdot u$

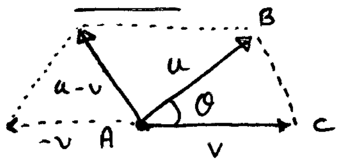
(2) $u \cdot u = u_1^2 + u_2^2 + u_3^2 = \|u\|^2$

(3) $u \cdot (v + w) = u \cdot v + u \cdot w$

The "cosine" theorem: $u \cdot V = \|u\| \, \|v\| \cos\theta$

angle between u and V

Proof :



$u = \overrightarrow{AB}$

$v = \overrightarrow{AC}$

$u - v = \overrightarrow{CB}$

In triangle ABC :

$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AC|\cos\theta$$

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta$$

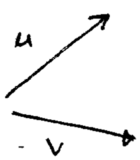
$$(u-v)\cdot(u-v) = u\cdot u + v\cdot v - 2\|u\|\|v\|\cos\theta$$

$$\cancel{u\cdot u} - u\cdot v - v\cdot u - \cancel{v\cdot v} = \cancel{u\cdot u} + \cancel{v\cdot v} - 2\|u\|\|v\|\cos\theta$$

$$\cancel{-}\cancel{2}u\cdot v = \cancel{-}\cancel{2}\|u\|\|v\|\cos\theta$$

$$u\cdot v = \|u\|\|v\|\cos\theta$$

Geometrical Interpretation of dot product



In case : $\boxed{\theta = \pi/2}$ (that is u and v are perpendicular)

$\cos\theta = 0 \quad \therefore \quad \|u\|\|v\|\cos\theta = 0$

$$\boxed{u\cdot v = 0}$$

Cross Product between 2 vectors

$$u = \langle u_1, u_2, u_3\rangle = u_1 i + u_2 j + u_3 k$$

$$v = \langle v_1, v_2, v_3\rangle = v_1 i + v_2 j + v_3 k$$

DEF : $u \times v =$ " $\begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ " $= (u_2v_3 - u_3v_2)i \ \ldots$

$\ldots - (u_1v_3 - u_3v_1)j \ \ldots$

$\ldots + (u_1v_2 - u_2v_1)k$

Properties : (1) $u \times u = 0$

(2) $u \times v$ = a vector perpendicular to both u and v (in fact, the "right hand rule" applies)

(3) $\|u \times v\| = \|u\|\|v\| \sin\theta$ ← angle between u and v

$u = \langle u_1, u_2, u_3 \rangle$

$u \times v = \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle$

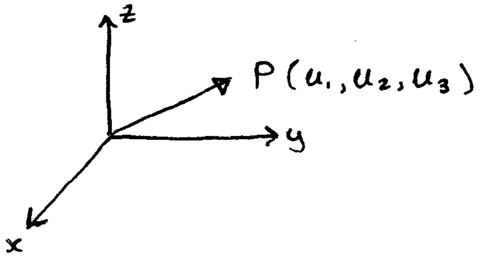
$u \cdot (u \times v) = u_1(u_2v_3 - u_3v_2) - u_2(u_1v_3 - u_3v_1) + \cdots$

$\cdots u_3(u_1v_2 - u_2v_1)$

$= u_1u_2v_3 - u_1u_3v_2 - u_2u_1v_3 + u_2u_3v_1 + u_3u_1v_2 - u_3u_2v_1$

$= 0$

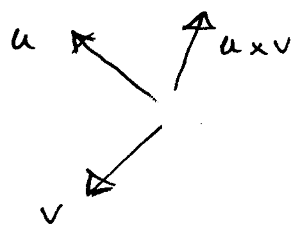
$u = \langle u_1, u_2, u_3 \rangle = u_1 \cdot i + u_2 \cdot j + u_3 k$



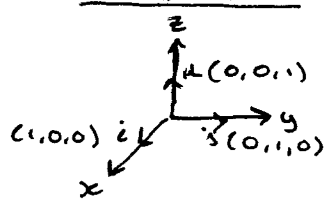
$V = \langle V_1, V_2, V_3 \rangle = V_1 i + V_2 j + V_3 k$

$$u \times V = \text{"}\begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\text{"}$$

$\Rightarrow (u_2 v_3 - v_2 u_3) i - (u_1 v_3 - v_1 u_3) j + (u_1 v_2 - v_1 u_2) k$



Properties:



(1) $u \times v = -v \times u$

(2) $u \times (v + w) = u \times v + u \times w$

(3) $\left.\begin{array}{l} i \times j = k \\ j \times k = i \\ k \times i = j \end{array}\right\}$ $\dots\dots$ $i\ j\ k\ i\ j$

Example: $u = \langle 2, 3, -1 \rangle$

$v = \langle 1, -1, 0 \rangle$

$w = \langle 7, 3, 2 \rangle$

Compute $u \cdot (v \times w)$

Sol: $v \times w = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{vmatrix} = (-2 - 0)i - (2 - 0)j + (3 - (-7))k$

$= -2i - 2j + 10k$

$= \langle -2, -2, 10 \rangle$

$u = \langle 2, 3, -1 \rangle$

$v \times w = \langle -2, -2, 10 \rangle$

$u \cdot (v \times w) = 2(-2) + 3(-2) + (-1)(10)$

$= -4 - 6 + 10 = \underline{\underline{0}}$

Geometrical interpretation for example:

- $v \times w$ is perpendicular to u
- $v \times w$ is perpendicular to both v and w

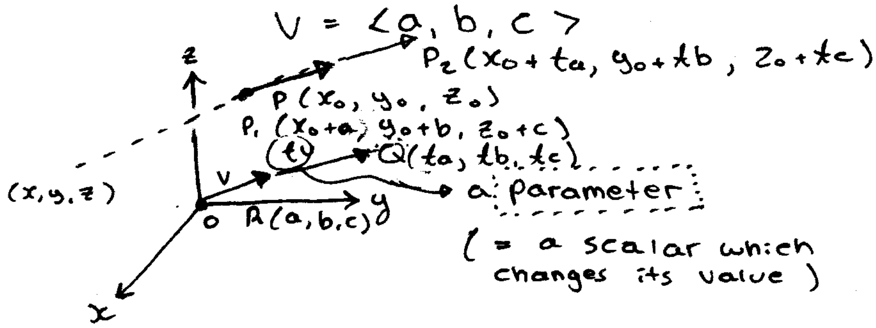
$\rightarrow$ $v \times w$ is perpendicular to u, v and w

$\rightarrow$ vectors u, v, w are coplane

(are in the same plane)

Equations of lines and Planes in $\mathbb{R}^3$

Lines : Setting: we will find the equations of a line that passes through a given point $P(x_0, y_0, z_0)$ and has a given direction $V = \langle a, b, c \rangle$



Equation of line :

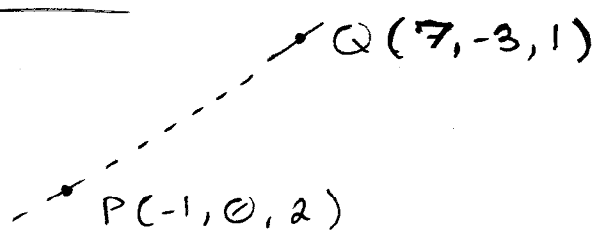
$$\boxed{\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}}$$

(Parametric Equation of the Line)

t = any scalar

Example Find equation of line that passes through $P(-1, 0, 2)$ and $Q(7, -3, 1)$

Solution:



- point $P(\overset{x_0}{-1}, \overset{y_0}{0}, \overset{z_0}{2})$
- direction $V = \overrightarrow{PQ}$

$= \langle 7-(-1), -3-0, 1-2 \rangle$

$= \langle \underset{a}{8}, \underset{b}{-3}, \underset{c}{-1} \rangle$

Eq. line :

$$① \begin{cases} x: & -1 + 8t \\ y: & 0 - 3t \\ z: & 2 - t \end{cases}$$

(t = parameters)

Solution #2 :

- point $Q(7, -3, 1)$
- direction $V = \overrightarrow{QP} = \langle -8, 3, 1 \rangle$

Eq. of line :

$$\left.\begin{aligned} x &= 7 - 8t \\ y &= -3 + 3t \\ z &= 1 + t \end{aligned}\right\} ②$$

Discussion :

Eq ① : $t = 0 \rightsquigarrow P(-1, 0, 2)$

$t = 1 \rightsquigarrow Q(7, -3, 1)$

Eq ② : $t = 0 \rightsquigarrow Q(7, -3, 1)$

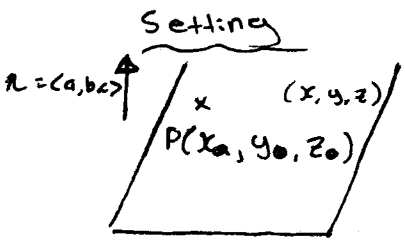
$t = 1 \rightsquigarrow P(-1, 0, 2)$

Remark : we are dealing with a RE-PARAMETRIZATION. Change $\boxed{t}$ in Eq ① into $\boxed{t-1}$

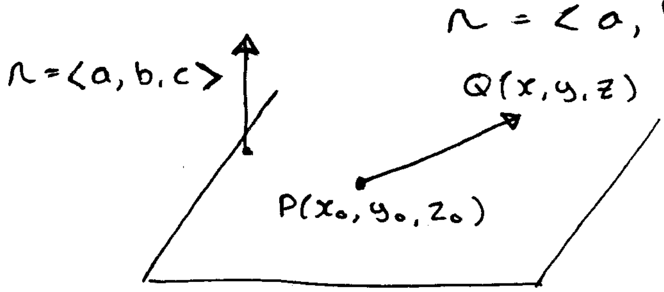
$$② = \left\{ \begin{array}{l} x = 7 - 8t \\ y = -3 + 3t \\ z = 1 + t \end{array} \right\} \longleftarrow \left\{ \begin{array}{l} x = 1 + 8(1-t) \\ y = 0 - 3(1-t) \\ z = 2 - (1-t) \end{array} \right.$$

Equation of Planes in $\mathbb{R}^3$

Setting



We will compute the eq. plane that passes through a given point $P(x_0, y_0, z_0)$ and has the vector $n = \langle a, b, c\rangle$ as a normal vector.



n is normal to plane

$\therefore$ n is perpendicular to $\overrightarrow{PQ}$

$\therefore$ $n \cdot \overrightarrow{PQ} = 0$

$n = \langle a, b, c\rangle$

$\overrightarrow{PQ} = \langle x - x_0, y - y_0, z - z_0\rangle$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

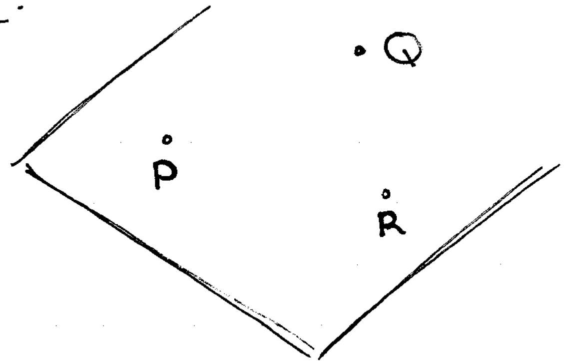
$$ax + by + cz - (ax_0 + by_0 + cz_0) = 0$$

$$\boxed{ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{\text{a number}}}$$

Components of normal vector to plane

Example Find the equation of the plane containing the points $P(-1,1,-2)$, $Q(0,1,3)$ and $R(2,-1,3)$

Solution:



$x_0\ y_0\ z_0$

- point $P(-1,1,-2)$
- normal vector

$n = \overrightarrow{PR} \times \overrightarrow{PQ}$

$\overrightarrow{PR} = \langle 1, 0, 5 \rangle$

$\overrightarrow{PQ} = \langle 3, -2, 5 \rangle$

$$n = \begin{vmatrix} i & j & k \\ 1 & 0 & 5 \\ 3 & -2 & 5 \end{vmatrix} \Rightarrow \begin{array}{ll} i: & 10 \\ j: & -[5-15] \\ k: & -2 \end{array} \Rightarrow 10i + 10j - 2k \Rightarrow \langle 10, 10, -2 \rangle$$

Eq. Plane:

$10(x-(-1)) + 10(y-1) - 2(z-(-2)) = 0$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$10x + 10 + 10y - 10 - 2z - 4 = 0$

$10x + 10y - 2z = 4$

Example Decide if the following planes $2x - 4y + 3z = 10$ and $-4x + 8y - 6z = 10$ are parallel.

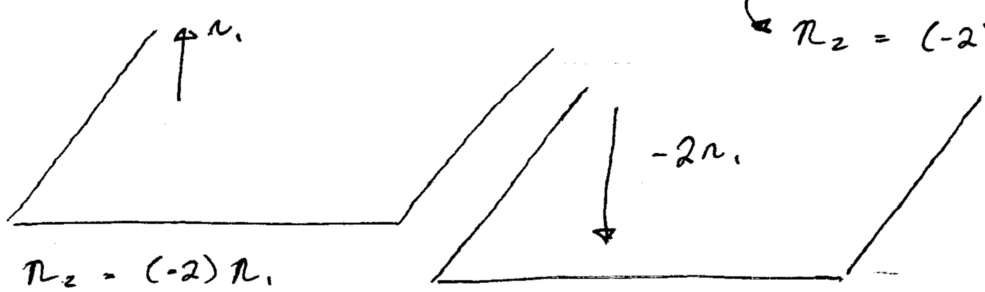
Solution: $2x - 4y + 3z = 10$ $\qquad -4x + 8y - 6z = 10$

1st plane

$n_1 = \langle 2, -4, 3 \rangle$ a normal vector

$n_2 = \langle -4, 8, -6 \rangle$ a normal vector

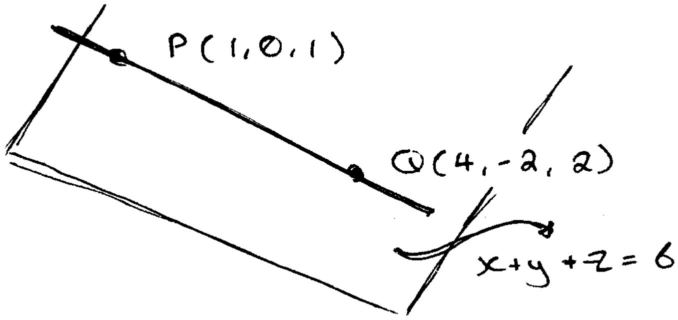
$n_2 = (-2) \cdot n_1$



Since $n_2 = (-2) n_1$ the planes are parallel

Ex. Where does the line through $(1, 0, 1)$ and $(4, -2, 2)$ intersect the plane $x+y+z=6$

Sol:



Line : point $P(\underset{x_0}{1}, \underset{y_0}{0}, \underset{z_0}{1})$

direction vector $v = \overrightarrow{PQ} = \langle 4-1, -2-0, 2-1 \rangle$

$= \langle \underset{a}{3}, \underset{b}{-2}, \underset{c}{1} \rangle$

Parametric eq. of line
$x = 1 + 3t$
$y = 0 - 2t$ where t = parameter
$z = 1 + t$

To find intersection ; Plane : $x + y + z = 6$

$\underbrace{(1+3t)}_{x} + \underbrace{(-2t)}_{y} + \underbrace{(1+t)}_{z} = 6$

$2 + 2t = 6$

$2t = 4 \Rightarrow t = 2$ ← intersection

Then $t=2$ corresponds

$x = 1 + 3(2) = 7$

$y = 0 - 2(2) = -4$

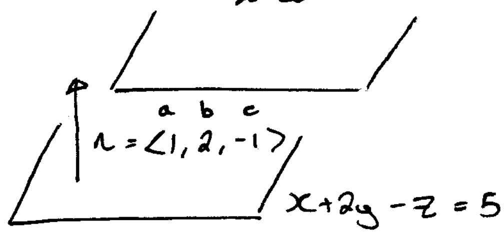
$z = 1 + 2 = 3$

Answer: $(7, -4, 3)$

✱ Ex (1) Find the equation of a plane that passes through the point $P(-1, 1, 2)$ and is parallel to $x + 2y - z = 5$.

(2) Find the equation of a line (any line) that passes through P and is parallel to the plane $x + 2y - z = 5$.

Sol (1): $P(\underset{x_0}{-1}, \underset{y_0}{1}, \underset{z_0}{2})$

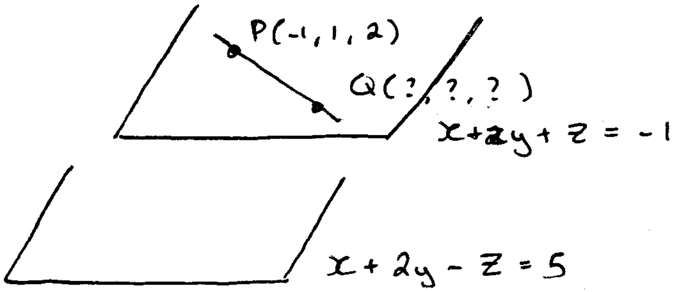


$1(x-(-1)) + 2(y-1) + 1(z-2) = 0$

$x+1 + 2y-2 + z-2 = 0$

$$\boxed{x + 2y + z = -1}$$

(2): $P(-1, 1, 2)$



Solution 1: we know
Plane $x + 2y - z = -1$
contains $P(-1, 1, 2)$
is parallel to $x + 2y + z = 5$
Find $Q(0, 0, 1)$ belonging to $x + 2y - z = -1$

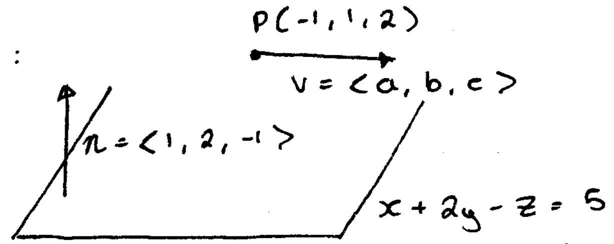
Now eq. of line through $P(-1,1,2)$ and $Q(0,0,1)$

- point $P(\underset{x_0}{-1}, \underset{y_0}{1}, \underset{z_0}{2})$
- vector $\overrightarrow{PQ} = \langle 0-(-1), 0-1, 1-2 \rangle$
$= \langle 1, -1, -1 \rangle$

$$\boxed{\begin{aligned} x &= -1 + t \\ y &= 1 - t \\ z &= 2 - t \end{aligned}}$$

t = parameter

Solution #2 :



$V = \langle 1, 0, 1 \rangle$

We are looking for a vector $V = \langle a, b, c \rangle$ such that V is perpendicular to $n = \langle 1, 2, -1 \rangle$

$V \cdot n = 0$ $\qquad$ $a = 1$

$a1 + b2 + c(-1) = 0$ $\qquad$ $b = 0$

$a + 2b - c = 0$ $\qquad$ $c = 1$

Eq: $\boxed{\begin{aligned} x &= -1 + t \\ y &= 1 \\ z &= 2 + t \end{aligned}}$

t = parameter

Ex : Find the line of intersection between planes $2x - 3y - 4z = -1$ and $x + 4y - 2z = 5$

Solution #1 : $\begin{cases} 2x - 3y + 4z = -1 \\ x + 4y - 2z = 5 \end{cases}$ $\qquad$ $\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$

Just take $z = t$ (changing)

$2x - 3y + 4t = -1$
$x + 4y - 2t = 5$ $\rightarrow$ $2x - 3y = -1 - 4t$
$x + 4y = 5 + 2t$

$$\begin{aligned} 2x - 3y &= -1 - 4t \\ - \quad 2x + 8y &= 10 + 4t \\ \hline 0 \; - 11y &= -11 - 8t \end{aligned}$$

$\rightarrow$ $\boxed{11y = 11 + 8t}$

Finding $x = -4y + 5 + 2t$
$= -4(1 + 8/11\, t) + 5 + 2t$
$= -4 - 32/11\, t + 5 + 2t$
$\Rightarrow$ $\boxed{x = 1 - 10/11\, t}$

Answer : $x = 1 - \frac{10}{11} t$
$y = 1 - \frac{8}{11} t$
$z = t$