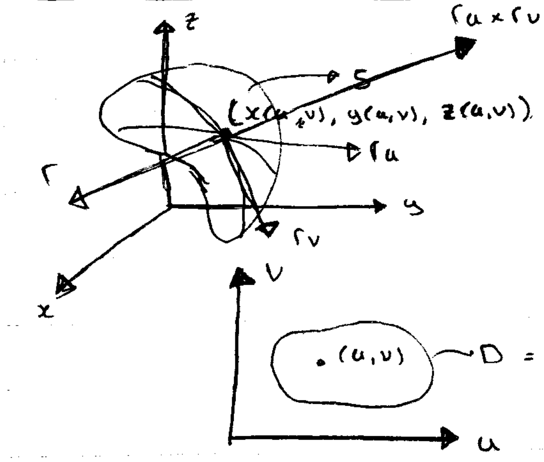
Normal Vectors to a Surface



S: $x = x(u,v)$

$y = y(u,v)$

$z = z(u,v)$

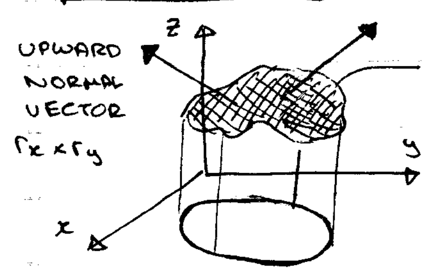
u, v = parameters

D = domain of u, v

$$r(u,v) = x(u,v)\,i + y(u,v)\,j + z(u,v)\,k$$

$$\left.\begin{aligned} r_u(\overset{\text{fixed}}{u,v}) &= \partial x/\partial u\, i + \partial y/\partial u\, j + \partial z/\partial u\, k \\ r_v(\underset{\text{fixed}}{u},v) &= \partial x/\partial v\, i + \partial y/\partial v\, j + \partial z/\partial v\, k \end{aligned}\right\} \text{vectors tangent to the surface}$$

$\boxed{\text{Example:}}$ S = <u>graph</u> of $g(x,y)$ $\big(z = g(x,y)\big)$



$z = g(x,y)$

S: $x = x$

$y = y$

$z = g(x,y)$

$$r(x,y) = x\,i + y\,j + g(x,y)\,k$$

$$r_x = 1\,i + 0\,j + \partial g/\partial x\, k$$

$$r_y = 0\,i + 1\,j + \partial g/\partial y\, k$$

$$r_x \times r_y = -\partial g/\partial x\, i - \partial g/\partial y\, j + 1\,k$$

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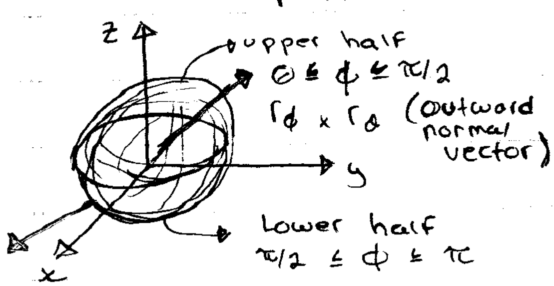
(2) S = Sphere of radius 3



S: $x = 3\sin\phi\cos\theta$

$y = 3\sin\phi\sin\theta$

$z = 3\cos\phi$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$$r(\phi, \theta) = 3\sin\phi\cos\theta\, i + 3\sin\phi\sin\theta\, j + 3\cos\phi\, k$$

$$r_\phi = \ldots$$

$$r_\theta = \ldots$$

$$r_\phi \times r_\theta = 9\sin^2\phi\cos\theta\, i + 9\sin^2\phi\sin\theta\, j + 9\sin\phi\cos\phi\, k$$

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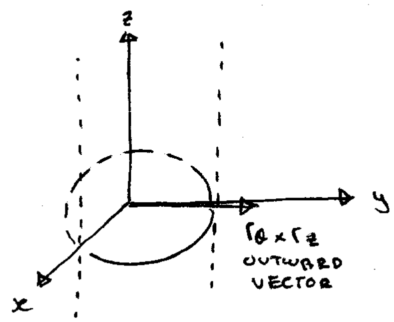
(3) S = cylinder $\quad x^2 + y^2 = 1$



$x = 1\cos\theta$

$y = 1\sin\theta$

$z = z$

θ, z = parameters

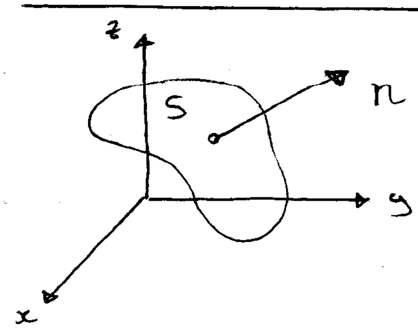
$$r(\theta, z) = \cos\theta\, i + \sin\theta\, j + z\, k$$

$$r_\theta =$$

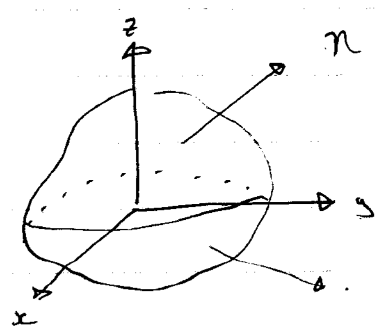
$$r_z =$$

$$r_\theta \times r_z = \underbrace{(\cos\theta)}_{\text{positive in first octant}} i + \underbrace{(\sin\theta)}_{\text{positive in first octant}} j + 0\, k$$

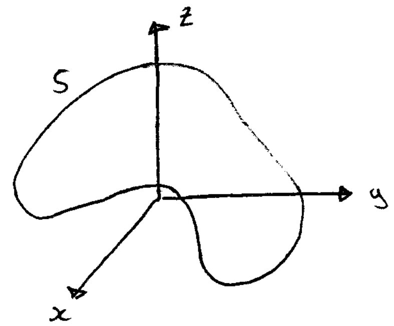
Orientation of a Surface



n = upward normal vector



n = outward normal vector



- F vector fields

$$F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$$

- Surface S : $x = x(u,v)$

$y = y(u,v)$

$z = z(u,v)$

dot product

DEF : " $\iint_S F(x,y,z) \cdot ds$ " $= \iint F(x(u,v), y(u,v), z(u,v)) \cdot (r_u \times r_v)\, du\, dv$

normal vector giving orientation of surface S

" Flux over surface "	$\iint_S F(x,y,z)\, ds = \iint F(x(u,v), y(u,v), z(u,v)) \cdot (r_u \times r_v)\, du\, dv$

$\boxed{\text{Ex}}$ Find the flux of the vector field :

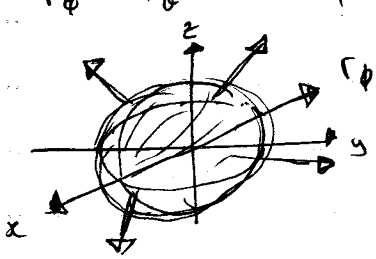
$$F(x,y,z) = z i + y j + 1 k$$

Across the unit sphere : $x^2 + y^2 + z^2 = 1$

Sol : $\iint_S F \cdot ds$

$$S : \left.\begin{aligned} x &= 1 \sin\phi \cos\theta \\ y &= 1 \sin\phi \sin\theta \\ z &= 1 \cos\phi \end{aligned}\right\} \quad \begin{aligned} 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{aligned}$$

$$r_\phi \times r_\theta = \sin^2\phi \cos\theta i + \sin^2\phi \sin\theta j + \sin\phi \cos\phi k$$



$r_\phi \times r_\theta$ = OUTWARD NORMAL VECTOR

F on the surface S

$\Rightarrow$ Now :

$$\iint_S F \cdot ds = \int_0^{2\pi}\int_0^{\pi} [\cos\phi i + \sin\phi \sin\theta j + 1 k] \cdots$$

$$\cdots \cdot [\sin^2\phi \cos\theta i + \sin^2\phi \sin\theta j + \sin\phi\cos\phi k]\, d\phi\, d\theta$$

normal vector

$$= \int_0^{2\pi}\int_0^{\pi} \sin^2\phi \cos\phi \cos\theta + \sin^3\phi \sin^2\theta + \sin\phi \cos\phi \; d\phi\, d\theta$$

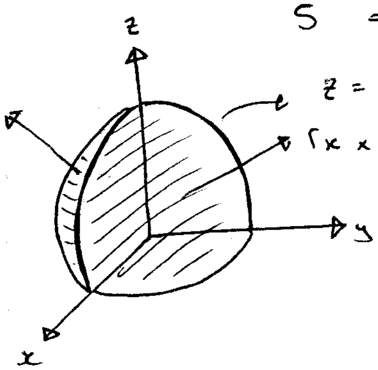
$u = \sin\phi$ $\quad$ $(1-\cos^2\phi)\cos\phi$ $\quad$ $u = \sin\phi$

$du = \cos\phi\, d\phi$ $\quad$ $u = \cos\phi$ $\quad$ $du = \cos\phi\, d\phi$

$du = -\sin\phi\, d\phi$

Ex. Find $\iint_S F \cdot ds$ where $F(x,y,z) = y\,i + x\,j + z\,k$

S = part of the paraboloid $z = 1 - x^2 - y^2$ bounded by $z = 0$



$$S: \quad x = x, \quad y = y, \quad z = \underbrace{1 - x^2 - y^2}_{g(x,y)}$$

$$r_x \times r_y = -\partial g/\partial x\, i - \partial g/\partial y\, j + 1k$$
$$= 2x \cdot i + 2y\, j + 1k$$

(x,y) in D = disc or radius 1

$$\iint_S F \cdot ds = \iint_D \underbrace{y i + x j + (1-x^2-y^2)k}_{F \text{ on surface } S} \cdot \underbrace{(2x i + 2y j + 1k)}_{\text{normal vector}} \, dA$$

$$= \iint_D [(2xy + 2xy + (1 - x^2 - y^2)] \, dA$$

$\hookrightarrow$ polar coordinates extra term

$$\Rightarrow \iint (4r^2 \cos\theta \sin\theta + 1 - r^2) \cdot r \;\; dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^1 4r^3 \cos\theta \sin\theta + r - r^3 = \ldots\ldots$$

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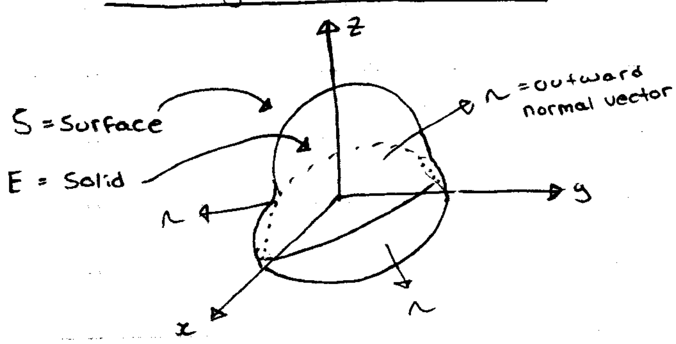
Operations	Input	Output
Gradient	Scalar Function $f(x,y,z)$	A vector Field $\nabla f(x,y,z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$
Divergence	Vector Field $F(x,y,z) = Pi + Qj + Rk$	A scalar Function $\text{div} F = \frac{\partial P}{\partial x} i + \frac{\partial Q}{\partial y} j + \frac{\partial R}{\partial z} k$
Curl	Vector Field $F(x,y,z) = Pi + Qj + Rk$	Curl (F)

$$\text{Curl}(F) = \text{``}\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}\text{''} \Rightarrow \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) i - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) j \dots$$

$$\dots + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k$$

Divergence Theorem



- vector Field
- S = Surface that encloses a Solid E

Then, $\iint_S F \cdot ds$

$$= \iiint_E \underbrace{\text{div} F}_{\text{Scalar Function}} \, dV$$

flux of F ← vector field

$\iint_S F \cdot ds$

$\int_a^b f(x)\,dx = F(b) - F(a)$

Here, S is oriented by it's OUTWARD normal vector

Ex: Evaluate the Flux of the vector Field

$$F(x,y,z) = \underbrace{xy}_{P}\,i + \underbrace{(y^2 + e^{xz^2})}_{Q}\,j + \underbrace{\sin(xy)}_{R}\,k$$

Over the Surface: S = the surface enclosing the solid E

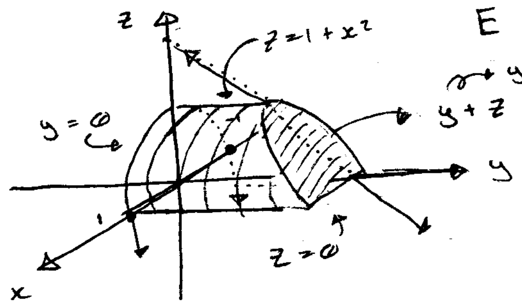
E = Solid bounded by $z = 1 - x^2$

$z = 0$

$y = 0$

$y + z = 2$

$z = 2 - y$



$$\underset{S}{\iint} F \cdot ds \underset{\text{divergence theorem}}{=} \underset{E}{\iiint} \underbrace{\operatorname{div} \mathbf{F}}_{\text{Scalar Function}}\, dV$$

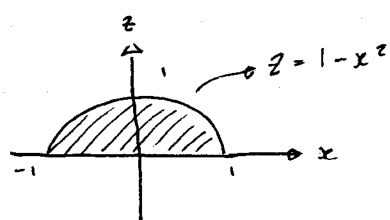
$$\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= y + 2y + 0$$

$$= 3y$$

Now: $\iiint_E 3y\, dV$

$E = \{ (x,y,z) : \; 0 \le y \le 2 - z, \; (x,z) \text{ in } D \}$



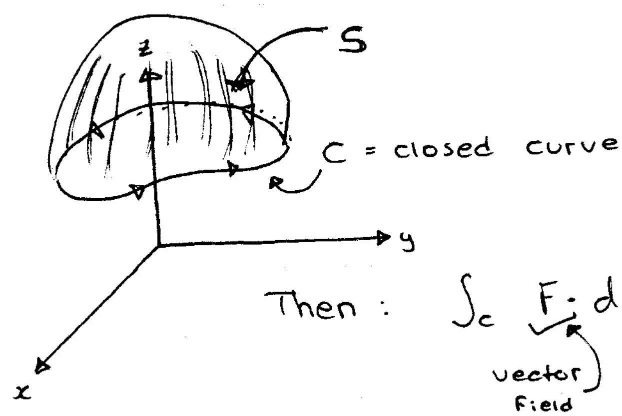
$\{ 0 \le z \le 1 - x^2, \; -1 \le x \le 1 \}$

$$= \int_{-1}^{1} \left[\int_0^{1-x^2} \left(\int_0^{2-z} 3y \, dy \right) dz \right] dx$$

$$\Rightarrow \int_{-1}^{1} \int_0^{1-x^2} (3/2) \left[y^2 \Big|_{y=0}^{y=2-z} \right] dz\, dx \quad \ldots$$

Stoke's Theorem

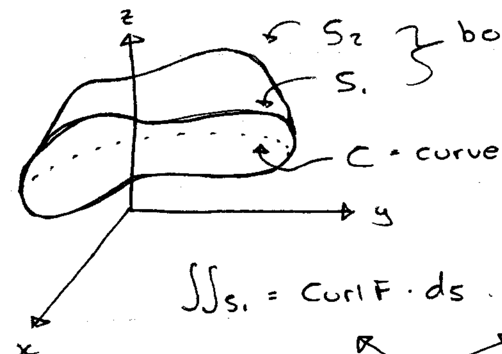
- Vector Field $F(x, y, z) = P(x,y,z)\,i + Q(x,y,z)\,j + R(x,y,z)\,k$
- Surface S which has a closed curve C as its boundary.



Then : $\displaystyle \int_C \underbrace{F}_{\text{vector field}} \cdot dr = \iint_S \underbrace{\text{curl} F}_{\text{vector field}} \cdot ds$

Here, the "right-hand rule" applies

Remark:

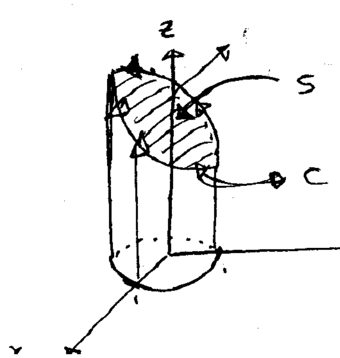


$$\underbrace{\iint_{S_1} \text{Curl} F \cdot ds = \int_C F \cdot dr = \iint_{S_2} \text{Curl} F \cdot ds}_{\text{independence of surface for flux}}$$

Ex. Evaluate $\int_C F \cdot dr$ where

$F(x, y, z) = -y^2\,i + x\,j + z^2\,k$

C = Curve of intersection between $x^2 + y^2 = 1$ and $y + z = 2$, with counter-clockwise orientation when viewed from above.



$x = r\cos\theta \qquad r(\theta) = \ldots$

$y = r\sin\theta \qquad r'(\theta) = \ldots$

$z = 2 - \underbrace{\sin\theta}_{y}$

$0 \leq \theta \leq 2\pi$

Solution #2 (Using Stoke's Theorem)

$$\int_c F\cdot dr = \iint_S \underbrace{\text{curl } F}_{\hookrightarrow \text{ vector field}} \cdot ds$$

Where S = Surface : $z = 2 - y$: oriented with upward normal vector

S: $x = x$

$y = y$

(as a graph) $z = 2 - y$ $[g(x,y)]$

x, y = parameters

(x,y) in D = disc of radius 1

$r(x,y) = \ldots$

$r_x \times r_y = -\partial g/\partial x \, i - \partial g/\partial y \, j + 1k$ → UPWARD NORMAL VECTOR.

$= j + k$ → UPWARD

Vector Field : curl F $= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) i - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k$$

For us, $F(x,y,z) = \underbrace{-y^2}_{P}\, i + \underbrace{x}_{Q}\, j + \underbrace{z^2}_{R}\, k$

$\text{Curl } F = 0 i + 0 j + (1 + 2y) k = (1+2y) k$

Finally, $\iint_S \text{curl } F \cdot ds = \iint \underbrace{(1+2y)k}_{\text{curl } F} \cdot \underbrace{j + k}_{r_x \times r_y} \, dx\, dy$

$= \iint_D (1+2y)\, dA$ → polar coord.

- No Stoke's Thm on Final.
  ↳ divergence will be asked