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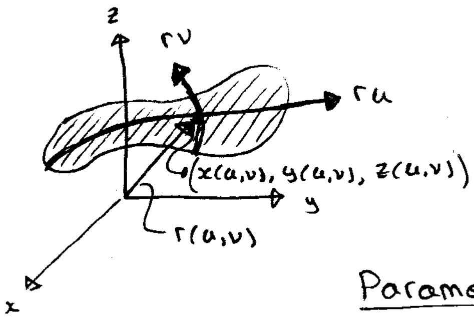
Surfaces



S : $x = x(u,v)$

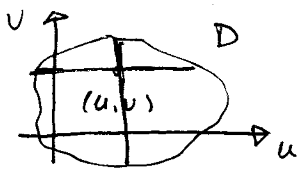
$y = y(u,v)$

$z = z(u,v)$

u, v = parameters

(u,v) in D = domain of Parameters

Parameters



Vector Valued Function

$$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$$

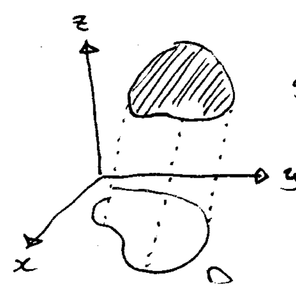
$$r_u = \frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j + \frac{\partial z}{\partial u} k$$

$$r_v = \frac{\partial x}{\partial v} i + \frac{\partial y}{\partial v} j + \frac{\partial z}{\partial v} k$$

Remark : r_u and r_v will give us the tangent plane to S.

Surfaces	Curves
• computation of surface area	• computations of arc length
• Surface integral for scalar functions	• line integral for scalar function

Special case : S = graph of $z = f(x,y)$



S: $z = g(x,y)$

$x = x$

$y = y$

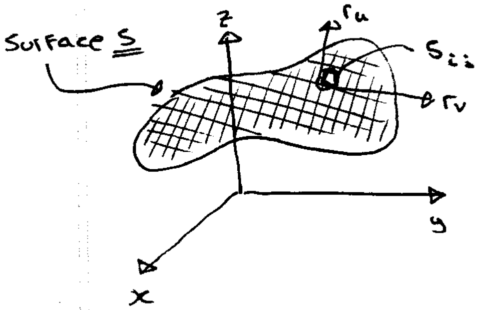
$z = g(x,y)$

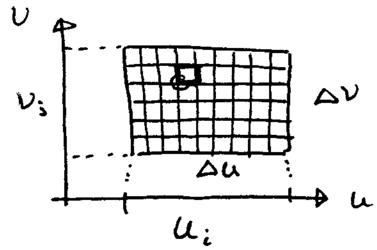
x, y = parameters

(x,y) in D = domain of parameters

$$r(x,y) = xi + yj + g(x,y)k$$

Computation of Surface Area





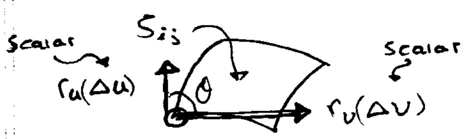
$x = x(u, v)$

$y = y(u, v)$

$z = z(u, v)$

$$\text{Area of } S = \lim_{n \to \infty} \sum_{i,j=1}^{n} \text{area of patch } S_{ij} = \boxed{\iint_D \|r_u \times r_v\| \, du \, dv}$$

D ← domain in u, v



$$S_{ij} \cong \text{area of parallelogram}$$
$$= \underbrace{\|r_v(\Delta v)\|}_{\text{base}} \; \underbrace{\|r_u(\Delta u\| \sin\theta}_{\text{height}}$$

$$= \| r_u(\Delta u) \times r_v(\Delta v)\|$$
$$= \|r_u \times r_v\| \Delta u \Delta v$$

Area of S =
$$= \iint_D \|r_u \times r_v\| \, du \, dv$$

Special case : S = graph of $z = g(x, y)$

$x = x$

$y = y$

$z = g(x, y)$

(x, y) in D

$$r(x, y) = x i + y j + g(x, y) k$$
$$r_x = 1 i + \emptyset j + \frac{\partial g}{\partial x} k$$
$$r_y = \emptyset i + 1 j + \frac{\partial g}{\partial y} k$$
$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & \emptyset & \frac{\partial g}{\partial x} \\ \emptyset & 1 & \frac{\partial g}{\partial y} \end{vmatrix}$$
$$\rightarrow -\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} j + k$$

$$\|r_x \times r_y\| = \sqrt{1 + \left[\frac{\partial g}{\partial x}\right]^2 + \left[\frac{\partial g}{\partial y}\right]^2}$$

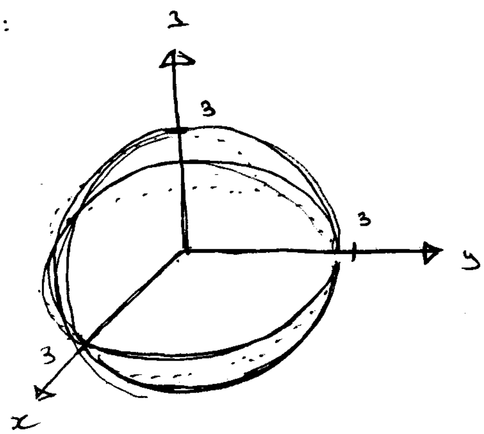
$$\text{Area of } S = \iint_D \|r_x \times r_y\| \, dA = \iint_D \sqrt{1 + \left[\frac{\partial g}{\partial x}\right]^2 + \left[\frac{\partial g}{\partial y}\right]^2} \, dA$$

Special case : S = graph of $z = g(x, y)$

$$\text{Area} = \iint_D \sqrt{1 + [\partial g/\partial x]^2 + [\partial g/\partial y]^2} \, dA$$

Ex : Compute the Surface area of a sphere of radius 3.

Sol :



$$x = 3\sin\phi\cos\theta$$
$$y = 3\sin\phi\sin\phi$$
$$z = 3\cos\phi$$
$$\phi, \theta = \text{parameters}$$

$$0 \le \theta \le 2\pi$$
$$0 \le \phi \le \pi$$

$$r(\phi,\theta) = \underbrace{3\sin\phi\cos\theta}\, i + \underbrace{3\sin\phi\sin\theta}\, j + \underbrace{3\cos\phi}\, k$$

$$r_\phi = 3\cos\phi\cos\theta\, i + 3\cos\phi\sin\theta\, j - 3\sin\phi\, k$$

$$r_\theta = -3\sin\phi\sin\theta\, i + 3\sin\phi\cos\theta\, j + 0\, k$$

$$r_\phi \times r_\theta = \begin{vmatrix} i & j & k \\ 3\cos\phi\cos\theta & 3\cos\phi\sin\theta & -3\sin\phi \\ -3\sin\phi\sin\theta & 3\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= 9\sin^2\phi\cos\theta\, i + 9\sin^2\phi\sin\theta\, j + (9\sin\phi\cos\phi\cos^2\theta + 9\sin\phi\cos\phi\cos^2\theta) k$$

$$r_\phi \times r_\theta = 9\sin^2\phi\cos\theta\, i + 9\sin^2\phi\sin^2\theta\, j + 9\sin\phi\cos\phi\, k$$

$$r_\phi \times r_\theta = \sqrt{81\sin^4\phi\cos^2\theta + 81\sin^4\phi\sin^2\theta + 81\sin^2\phi\cos^2\phi}$$

$$= \sqrt{81\sin^4\phi(\cos^2\theta + \sin^2\theta) + 81\sin^2\phi\cos^2\theta}$$

$$\Rightarrow \sqrt{81\sin^4\phi + 81\sin^2\phi\cos^2\theta}$$

$$\Rightarrow \sqrt{81\sin^2\phi(\sin^2\phi + \cos^2\theta)} \Rightarrow \sqrt{81\sin^2\phi} = 9\sin\phi$$

radius squared

Surface area

$$= \int_0^{2\pi}\int_0^{\pi} \| r_\phi \times r_\theta \| \, d\phi\, d\theta$$

$$= \int_0^{2\pi}\left[\int_0^{\pi} 9\sin\phi \, d\phi\right] d\theta \Rightarrow \int_0^{2\pi} -9\cos\phi \Big|_{\phi=0}^{\phi=\pi} d\theta$$

$$\ldots = 36\pi = (4\pi(\text{radius})^2)$$

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Area of Surface S =

$= \iint_D \|r_u \times r_v\| \, du\, dv$

$D \leftarrow$ domain of u, v

Special case

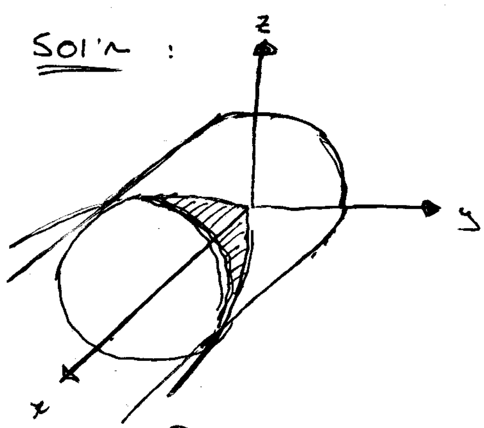
S = graph of $g(x,y)$

$[\, z = g(x,y) \,]$

$$\text{Area} = \iint \sqrt{1 + \left[\frac{\partial g}{\partial x}\right]^2 + \left[\frac{\partial g}{\partial y}\right]^2}\, dx\, dy$$

Ex. Compute the surface area of the part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$.

Sol'n :



Parameterization #1

$x = y^2 + z^2$

$y = y$ (as a graph)

$z = z$

y, z as a parameter

(y, z) in D = disc of radius 3

Parameterization #2

$x = y^2 + z^2 = r^2$

$y = r\cos\theta$

$z = r\sin\theta$

(r, θ) = parameters

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

For #1 : Surface area (as a graph) $= \iint_{D \,=\, \text{disc of radius } 3} \sqrt{1 + \left[\frac{\partial g}{\partial y}\right]^2 + \left[\frac{\partial g}{\partial x}\right]^2}\, dy\, dz$

$= \iint \sqrt{1 + (2y)^2 + (2z)^2}\, dy\, dz \longrightarrow$ change to polar coordinate

$\longrightarrow \int_0^{2\pi}\int_0^3 \sqrt{1 + 4r^2}\; \underbrace{r}_{\text{extra term}}\; dr\, d\theta$

For #2:

$$\underline{r}(r,\theta) = r^2 i + r\cos\theta\, j + r\sin\theta\, k$$

$$\underline{r}_r = 2r i + \cos\theta\, j + \sin\theta\, k$$

$$\underline{r}_\theta = 0 - r\sin\theta\, j + r\cos\theta\, k$$

$$\underline{r}_r \times \underline{r}_\theta = \begin{vmatrix} i & j & k \\ 2r & \cos\theta & \sin\theta \\ 0 & -r\sin\theta & r\cos\theta \end{vmatrix} \longrightarrow \begin{array}{l} i : r\cos^2\theta + r\sin^2\theta \\ j : -[2r^2\cos\theta] \\ k : -2r^2\sin\theta \end{array}$$

$$[r\cos^2\theta + r\sin^2\theta] i - 2r^2\cos\theta\, j - 2r^2\sin\theta\, k$$

$$\|\underline{r}_r \times \underline{r}_\theta\| = \sqrt{r^2 + 4r^4\cos^2\theta + 4r^4\sin^2\theta}$$

$$= \sqrt{r^2 + 4r^4} \Rightarrow \sqrt{r^2(1+4r^2)}$$

$$= r\sqrt{1+4r^2}$$

$$\text{Surface area} = \int_0^{2\pi}\int_0^3 r\sqrt{1+4r^2}\, dr\, d\theta$$

Surface Integral of <u>Scalar Functions</u>

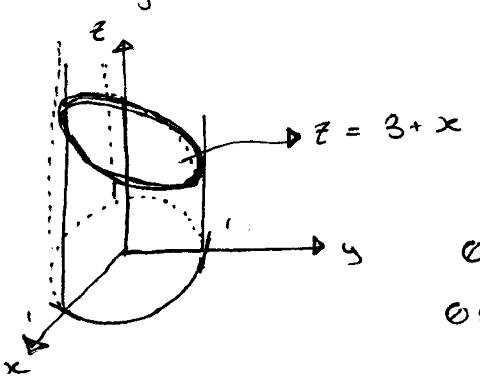
DEFN: "$\iint_S f(x,y,z)\, dS = \iint \underbrace{f(r(u,v))}_{\text{density}} \underbrace{\|r_u \times r_v\|\, du\, dv}_{\text{surface area part}}$

<u>Special Case</u> : S = graph of $g(x,y)$

$[z = g(x,y)]$

$$\iint_S f(x,y,z)\, dS = \iint_D \underbrace{f(x,y,g(x,y))}_{\text{density}} \underbrace{\sqrt{1 + \left[\frac{\partial g}{\partial x}\right]^2 + \left[\frac{\partial g}{\partial y}\right]^2}}_{\text{surface area part}}$$

$\boxed{\text{Ex:}}$ <u>Evaluate</u> : $\iint_S z\, ds$ where S = part of the cylinder $x^2+y^2=1$ between the planes $z=0$ and $z=3+x$



$S = \quad x = r\cos\theta = 1\cos\theta$

$\quad\quad y = r\sin\theta = 1\sin\theta$

$\quad\quad z = z$

$0 \le \theta \le 2\pi$

$0 \le z \le 3 + \underbrace{\cos\theta}_{x}$

$$r(\theta, z) = \cos\theta\, i + \sin\theta\, j + z\, k$$

$$r_\theta = -\sin\theta\, i + \cos\theta\, j + \emptyset\, k$$

$$r_z = \emptyset + \emptyset + 1k$$

$$r_\theta \times r_z = \begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & \emptyset \\ \emptyset & \emptyset & 1 \end{vmatrix} \Rightarrow \begin{aligned} i &= \cos\theta \\ j &= -[-\sin\theta] \\ k &= \emptyset \end{aligned}$$

$$\| r_\theta \times r_z \| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\Rightarrow \iint_S \underbrace{z}_{f(x,y,z)} \, ds \Rightarrow \iint \underbrace{z}_{\text{density}} \underbrace{\| r_\theta \times r_z \| \, dz\, d\theta}_{\text{surface area part}}$$

$$= \int_\emptyset^{2\pi} \int_\emptyset^{(3+\cos\theta)} z\,(1)\, dz\, d\theta = \int_\emptyset^{2\pi} \frac{z^2}{2} \Big|_{z=\emptyset}^{z=(3+\cos\theta)} d\theta$$

$$= \int_\emptyset^{2\pi} \frac{(3+\cos\theta)^2}{2} = \frac{1}{2}\int_\emptyset^{2\pi} (3+\cos\theta)^2 \ldots$$

Surface Integrals of Vector Fields

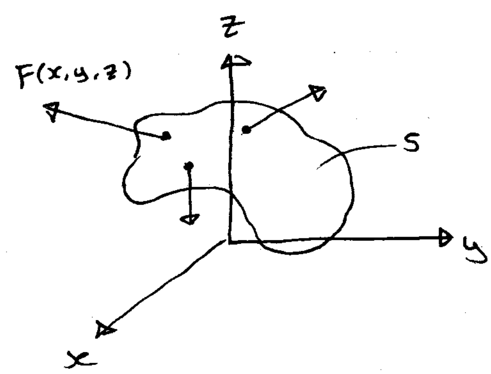
$F(x, y, z) = $ 3-dim. vector field

$\hookrightarrow P(x,y,z)\, i + Q(x,y,z)\, j + R(x,y,z)\, k$

Surface S : $x = x(u, v, w)$

$y = y(u, v, w)$

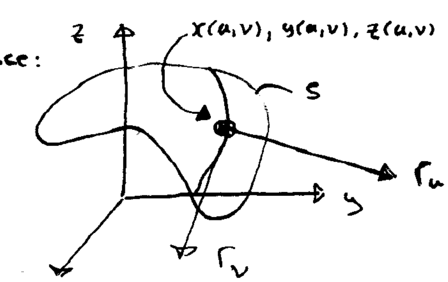
$z = z(u, v, w)$



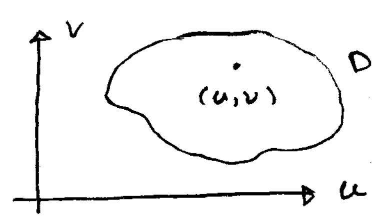
Normal Vectors to a surface S

$S: r(u,v) = x(u,v)\, i + y(u,v)\, j + z(u,v)\, k$

Surface:

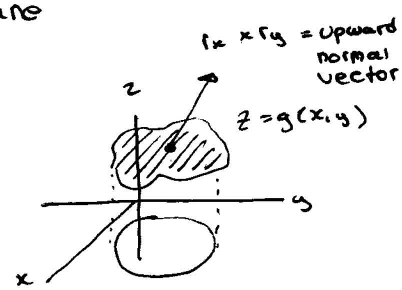


Parameters



Remark : r_u, r_v gives us the tangent plane

$\boxed{r_u \times r_v}$ = normal vector to surface



Ex : S = graph of $g(x,y)$

$$[\ z = g(x,y)\ ]$$

S: $x = x$

$y = y$

$z = g(x,y)$

$$r(x,y) = x\,i + y\,j + g(x,y)\,k$$

$$r_x = 1\,i + 0\,j + \frac{\partial g}{\partial x}\cdot k$$

$$r_y = 0 + 1\,j + \frac{\partial g}{\partial y}\,k$$

$$r_x \times r_y = -\frac{\partial g}{\partial x}\,i - \frac{\partial g}{\partial y} + 1k$$

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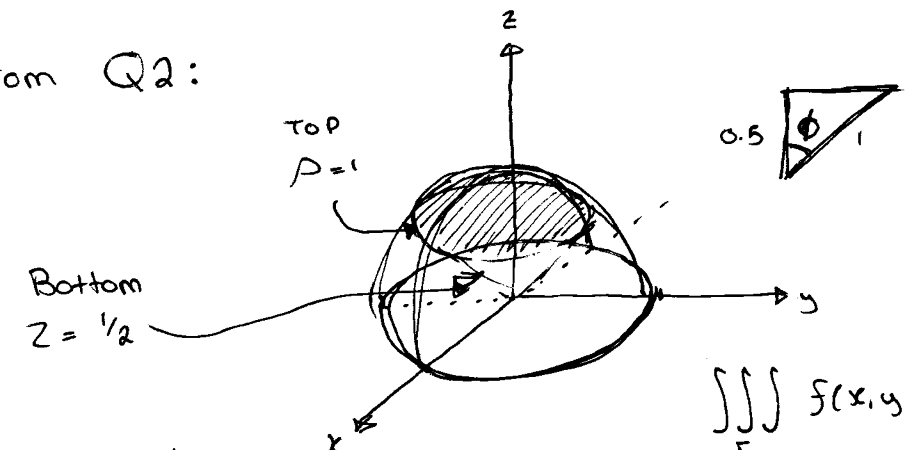
$$\int_{(-10,-9)}^{(-8,-2)} P(x,y)\,dx + Q(x,y)\,dy$$

Check : $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ (if equal, conservative)

$= \int_C F \cdot dr$, where $F(x,y) = P(x,y)\,i + Q(x,y)\,j$

$F = \nabla \tilde{f} \Rightarrow \tilde{f}(-8,-2) - \tilde{f}(-10,-9)$

From Q2:



$\cos\phi = \frac{1}{2}$

$\phi = \pi/3$

$\iiint_E f(x,y,z)\,dV$

$\rho\cos\phi = \frac{1}{2}x$

$\rho = \dfrac{1}{2\cos\phi}$

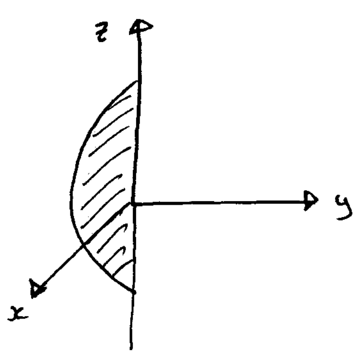
$$E = \{(\rho,\phi,\theta) \Rightarrow \left.\begin{array}{l} \frac{1}{2\cos\phi} \le \rho \le 1 \\ 0 \le \phi \le \frac{\pi}{3} \\ 0 \le \theta \le 2\pi \end{array}\right\}$$

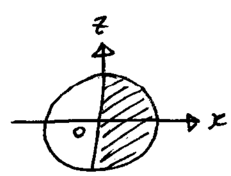
$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\theta$

From Q3 :





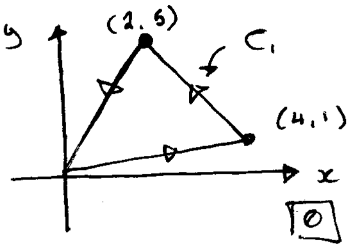
$-\sqrt{9-x^2-z^2} \le y \le \sqrt{9-x^2-z^2}$

$-\sqrt{9-x^2} \le z \le \sqrt{9-x^2}$

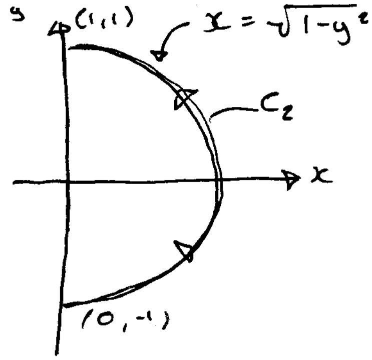
$\le x \le 3$

#16 $\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy$

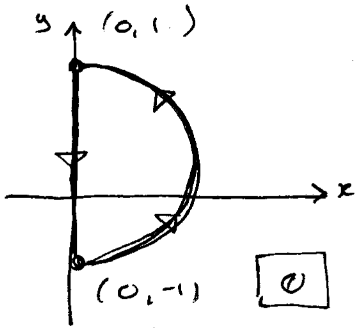
(1)



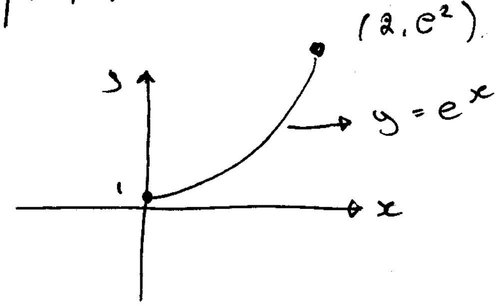
(2)



(3)



(4)



$\rightarrow \int_C F(x,y)\cdot dr$ for $F(x,y) = \underbrace{(2x - 3y + 1)\,i}_{P(x)} - \underbrace{(3x + y - 5)\,j}_{Q(x)}$

$\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ $\rightsquigarrow$ F is conservative — FTLI, indep.

$\quad -3 \qquad -3$

$\int_C F\cdot dr = F\nabla f \rightsquigarrow F(\text{end point}) - F(\text{initial point})$

$\rightarrow F(x,y) = (2x - 3y + 1)\,i - (3x + y - 5)\,j$

$\nabla(x,y) = \dfrac{\partial f}{\partial x}\,i + \dfrac{\partial f}{\partial y}\,j$

$\dfrac{\partial f}{\partial x} = 2x + 3y + 1 \longrightarrow f(x,y)\int(2x + 3y + 1)\,dx \rightarrow x^2 - \underline{3yx} + x + g(y)$

$\dfrac{\partial f}{\partial y} = 3x + y - 5 \longrightarrow f(x,y)\int(3x + y - 5)\,dy \rightarrow \underline{3xy} + y^2 - 5y + h(x)$

where $h(x) = x^2 + x$, $g(y) = y^2 - 5y$

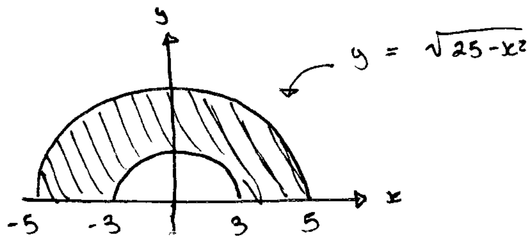
... etc.

15.4 (#20 - excluded from practice set)

#16 or $\int_C \underbrace{(y-x)}_{P(x,y)}\,dx + \underbrace{(2x-y)}_{Q(x,y)}\,dy \longrightarrow \int_C F(x,y)\cdot dr$

C = boundary of the region lying inside the semi-circle $y = \sqrt{25-x^2}$ and outside the semi-circle $y = \sqrt{9-x^2}$.

$F(x,y) = (y-x)\,i + (2x-y)\,j$



Green's Theorem $\iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$

$$= \iint_D (2-1)\,dA = \iint_D 1\,dA = \int_0^{\pi}\int_3^5 1 \cdot r\,dr\,d\theta$$