Nov. 13/18

Midterm:

Double integral to end of today's lecture

Vector Field 2-dimensional

$F(x,y) = P(x,y)\,i + Q(x,y)\,j$

Last Time: We say that F is conservative if we can find a scalar function $f(x,y)$ such that $F = \nabla f$

Operation	Input	Output
Gradient	Scalar Function $f(x,y)$	Vector Field $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$

(I) Fundamental Theorem of Line Integrals

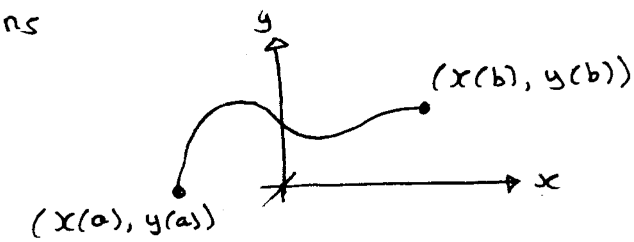
C = Curve in 2-dimensions

$F(x,y)$ = Vector Field

If f is conservative $(F = \nabla f)$

then:

$$\boxed{\int_C F \cdot dr = \ldots = \underbrace{f(x(b),y(b))}_{\text{end point}} - \underbrace{f(x(a),y(a))}_{\text{initial point}}}$$



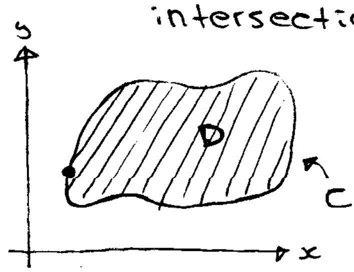
C : $x = x(t)$

$y = y(t)$

$a \le t \le b$

(II) Green's Theorem

C = closed curve in 2-dimensions (with no self-intersection)



$F(x,y) = P(x,y)\,i + Q(x,y)\,j$

Then $\int_C F \cdot dr = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$

(here C has counter-clockwise orientation)

Remarks

(1) Recognizing that a vector field is conservative

$F(x,y) = P(x,y)i + Q(x,y)j$

Condition : $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

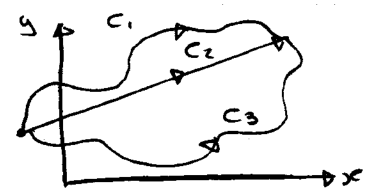
(2) Finding the potential function f such that

$F = \nabla f$

(3) If F is conservative, the Fundamental Thm of the integrals tells us that we have

Independence of Path :

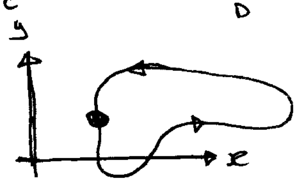
$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr = \int_{C_3} F \cdot dr$$



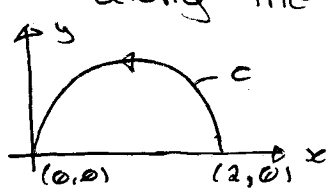
(4) When C is a closed curve and F is conservative we get, by Green's Thm :

$$\int_C F \cdot dr = \iint_D \underbrace{\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]}_{= \emptyset} dA = \emptyset$$

$\emptyset$ (F is conservative)



Ex: Compute the total work done to move a particle along the semi-circle C



under the action of :

$F(x,y) = (y^3 + 1)i + (3xy^2 + 1)j$

Solutions: #1 direct definition

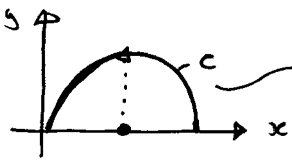
#2 FTLI

#3 Independence of Path

Solution #1 (direct def'n)

$\int_C F \cdot dr$

$C: \quad x-1 = \cos t \quad y = \sin t \quad \Rightarrow \quad \boxed{x = 1+\cos t,\ y = \sin t,\ 0 \le t \le \pi}$



$(x-1)^2 + y^2 = 1$

$r(t) = (1+\cos t)i + (\sin t)j$

$r'(t) = (-\sin t)i + (\cos t)j$

$\int_C F \cdot dr = \int_0^\pi \left[((\sin t)^3 + 1)i + (3(1+\cos t)(\sin^2 t + 1))j\right] \cdot (-\sin t\, i + \cos t\, j)\,dt$

$\Rightarrow \int_0^\pi -\sin t(\sin^3 t + 1) + \cos t(3\sin^2 t + 3\sin^2 t \cos t + 1)\,dt$

$\Rightarrow \int_0^\pi \left[-\sin^4 t - \sin t + 3\sin^3 t \cos t + 3\sin^2 t \cos^2 t + \cos t\right] dt$

= ... not pleasant

Solution #2 (FTFL)

Is $F(x,y) = \underbrace{(y^3+1)}_{P}i + \underbrace{(3xy^2+1)}_{Q}j$ conservative?

$\frac{\partial P}{\partial y} = 3y^2$

$\frac{\partial Q}{\partial x} = 3y^2$

thus, F is conservative

Find F such that $F = \nabla f$

$F(x,y) = (y^3+1)i + (3xy^2+1)j$

$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$

(A) $\frac{\partial f}{\partial x} = y^3 + 1$

(B) $\frac{\partial f}{\partial y} = 3xy^2 + 1$

(A) $\frac{\partial f}{\partial x} = y^3 + 1 \rightsquigarrow f(x,y) = \int (y^3+1)\,dx$ Fixed

$f(x,y) = y^3 x + x + \underbrace{g(y)}_{\text{constant}}$

(B) $\frac{\partial f}{\partial y} = 3xy^2 + 1 \rightsquigarrow f(x,y) = \int (3xy^2+1)\,dy$ Fixed

$f(x,y) = y^3 x + y + \underbrace{h(x)}_{\text{constant}}$

Take $g(y) = y$ and $h(x) = x$ and we get

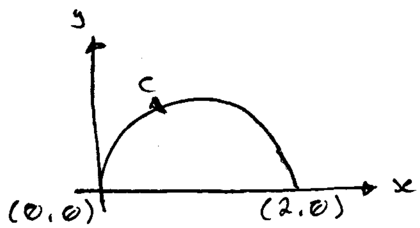
$$f(x,y) = y^3x + x + y$$

$$F = \nabla f$$

Fundamental Theorem of Line Integrals

$$\int_C F \cdot dr = \underset{\text{end point}}{f(0,0)} - \underset{\text{initial point}}{f(2,0)} = 0 - 2 = \boxed{-2}$$

Solution #3 (Independence of path)



$$F(x,y) = \underbrace{(y^3+1)}_{P} i + \underbrace{(3xy^2+1)}_{Q} j$$

$$\left.\begin{aligned} \partial P/\partial y &= 3y^2 \\ \partial Q/\partial x &= 3y^2 \end{aligned}\right\} \text{conservative}$$

$\Rightarrow$ Independence of path

$$\int_C F \cdot dr \qquad \int_{\tilde{C}} F \cdot dr$$

$$\tilde{C}: \quad \begin{aligned} x &= 2 - t \\ y &= 0 \\ 0 \le t &\le 2 \end{aligned}$$

$$r(t) = (2-t)i + (0)j$$

$$r'(t) = -1i + 0j$$

$$\int_{\tilde{C}} F \cdot dr = \int_0^2 [i + j] \cdot [-i + 0j]\, dt$$

$$\int_0^2 -1\, dt = \boxed{-2}$$

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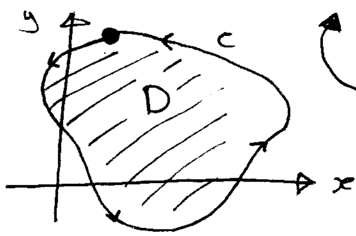
Green's Theorem

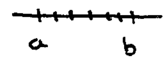
C = 2-dim curve, closed (with no self-intersection)

$F(x,y) = P(x,y)i + Q(x,y)j$



$\int_C P(x,y)dx + Q(x,y)dy \stackrel{=}{\leftrightarrow}$

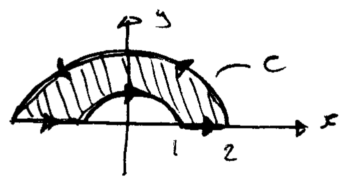
$$\oint_C F \cdot dr = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$



$$\int_a^b \underset{}{\overset{F(x)}{f(x)}}dx = F(b) - F(a)$$

$\boxed{\text{Ex:}}$ Compute $\oint_C \underbrace{(3y - e^{\sin x})}_{P(x,y)} dx + \underbrace{(7x + \sqrt{y^4+1})}_{Q(x,y)} dy$

where C = boundary of the region D in the upper half plane between $x^2+y^2 = 1$ and $x^2+y^2=4$



$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

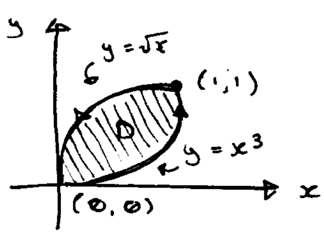
$$\Rightarrow \iint_D (7-3)dA = \iint_D 4\,dA$$

$$\xrightarrow[\text{Polar coord}]{} \int_0^{\pi}\int_1^2 \underset{\text{extra term}}{4r}\, dr\,d\theta = \int_0^{\pi} 2r^2 \Big|_{r=1}^{r=2} d\theta = \int_0^{\pi} (8-2)\,d\theta = 6\pi$$

cooler

$\boxed{\text{Ex:}}$ Compute $\int (y^3 dx + (x^3 + xy^2)) dy$

where C = path from $(0,0)$ to $(1,1)$ along the graph of $y = x^3$ and from $(1,1)$ to $(0,0)$ along $y = \sqrt{x}$



Sol'n: Green's Thm:

$$\int_C P(x,y)dx + Q(x,y)dy$$

$$\Rightarrow \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA = \iint (3x^2 + y^2) - 3y^2 \,] \,dA$$

$$\Rightarrow \int_0^1 \left[\int_{x^3}^{\sqrt{x}} (3x^2 - 2y^2)\, dy\right] dx$$

Surfaces and Surface Integrals

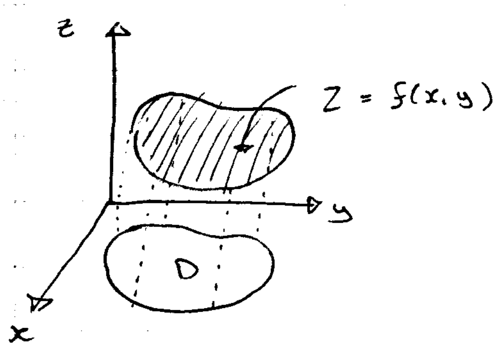
Overview:

Surfaces	Curves
• Parameterization of surfaces	• Parameterization of curves
• Surface area computation	• Arc length computation
• Surface integral of scalar functions	• Line integral of scalar function
• Surface integral for vector fields	• line integral for vector fields
→ 2 main results → Divergence Thm. → Stokes's thm.	→ 2 main results → Fundamental Thm. of Line Integrals (FTLI) → Green's Theorem

Parameterization of Surfaces

Until now, we have met the following special case of a surface: S = graph of $f(x,y)$

$$= \{(x,y,z) : z = f(x,y) \text{ with } (x,y) \text{ in } D\}$$



In general, we can parameterize a surface S in 3-dim as follows:

$$S = \{(x,y,z) : \begin{array}{l} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{array}$$

$$\begin{cases} \text{with } (u,v) = \text{parameters} \\ (u,v) \text{ in } D = \text{domain of parameters} \end{cases}$$

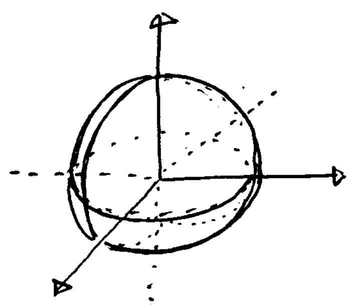
$\boxed{\text{Ex.}}$ Parameterize the following surfaces:

(1) Sphere of radius 1, centered at origin

(2) Surface of cylinder $x^2 + z^2 = 9$ enclosed by the planes $y = 0$, $y = 4$, $x = 0$

(3) Part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cone $z = \sqrt{x^2 + y^2}$

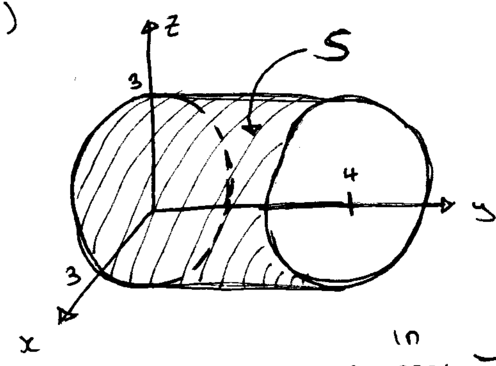
<u>Sol</u> : (1)



$$\left.\begin{aligned} x &= 1\sin\phi\cos\theta \\ y &= 1\sin\phi\sin\theta \\ z &= 1\cos\phi \end{aligned}\right\} \quad \begin{aligned} &\phi, \theta = \text{parameters} \\ &0 \le \theta \le 2\pi \\ &0 \le \phi \le \pi \end{aligned}$$

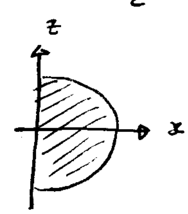
$$\text{in general} \left\{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}\right.$$

(2)

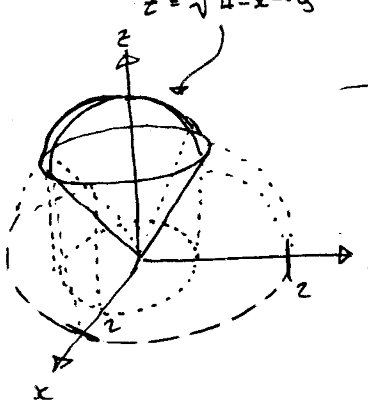


$$S: \left.\begin{aligned} x &= 3\cos\theta \\ y &= y \\ z &= 3\sin\theta \end{aligned}\right\} \quad \begin{aligned} &\theta, y = \text{parameters} \\ &0 \le y \le 4 \\ &-\tfrac{\pi}{2} \le \theta \le \pi/2 \end{aligned}$$

$$\text{in general} \left\{\begin{aligned} x &= r\cos\theta \\ y &= y \\ z &= r\sin\theta \end{aligned}\right.$$



(3)



(Solution #1 :) (spherical coord)

$$S: \left.\begin{aligned} x &= 2\sin\phi\cos\theta \\ y &= 2\sin\phi\sin\theta \\ z &= \end{aligned}\right\} \quad \begin{aligned} &\theta, \phi = \text{param.} \\ &0 \le \theta \le 2\pi \\ &0 \le \phi \le \pi/4 \end{aligned}$$

$$\text{in general} \left\{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}\right.$$

Solution #2 (as a graph)

$x = x$

$y = y$

$z = \sqrt{4 - x^2 - y^2}$

x, y = parameter

(x, y) in D = disc of radius $(?)$

Intersection :

$$\begin{cases} z = \sqrt{4 - x^2 - y^2} \\ z = \sqrt{x^2 + y^2} \end{cases}$$

$\sqrt{4 - x^2 - y^2} = \sqrt{x^2 + y^2}$

$4 - x^2 - y^2 = x^2 + y^2$

$4 = 2x^2 + 2y^2$

$2 = x^2 + y^2$

Solution #3 (using cylindrical coord)

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \text{in general}$$

S: $x = r\cos\theta$

$y = r\sin\theta$

$z = \sqrt{4 - \underbrace{r^2}_{(x^2+y^2)}}$ (hemisphere)

r, θ = parameters

$0 \le \theta \le 2\pi$

$0 \le r \le \sqrt{2}$

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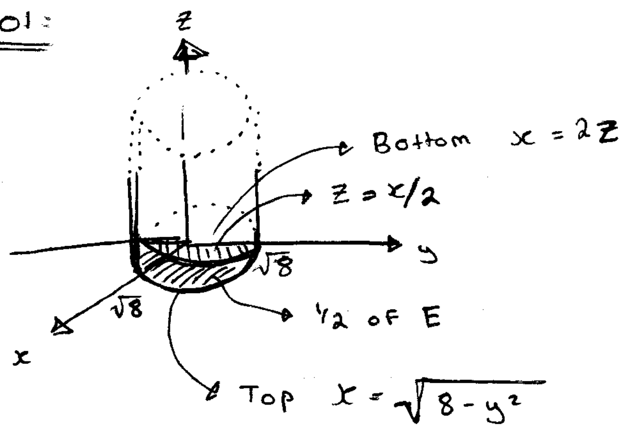
Ex: Let E be the solid enclosed by $8 = x^2 + y^2$ and by the planes $z = 0$, $z = x/2$

(E is the solid above xy-plane)

Set up, but do not evaluate, the volume of E as a triple integral in the order $dx\,dy\,dz$

Sol:



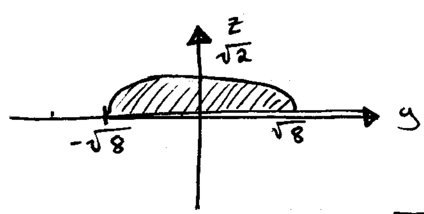
$$\text{Vol}(E) = \iiint_E 1\, dV = \int \int \int$$

$$\underset{\text{BOTTOM}}{2z} \le x \le \underset{\text{TOP}}{\sqrt{8 - y^2}}$$

(y, z) in D = domain in yz-plane }

↳ intersection between

$$2z = \sqrt{8 - y^2}$$
$$4z^2 = 8 - y^2$$
$$4z^2 + y^2 = 8 \quad \text{(ellipse)}$$



$$y^2 = 8 - 4z^2$$
$$y = \pm\sqrt{8 - 4z^2}$$

then $D \{ (y, z) : -\sqrt{8 - 4z^2} \le y \le \sqrt{8 - 4z^2}$

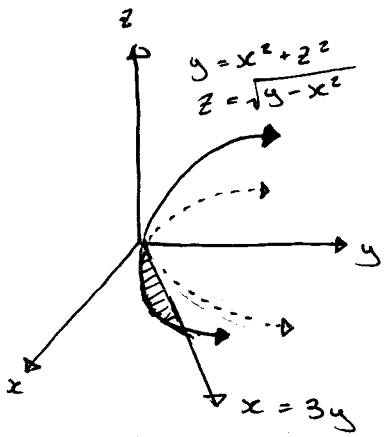
$$0 \le z \le \sqrt{2}$$

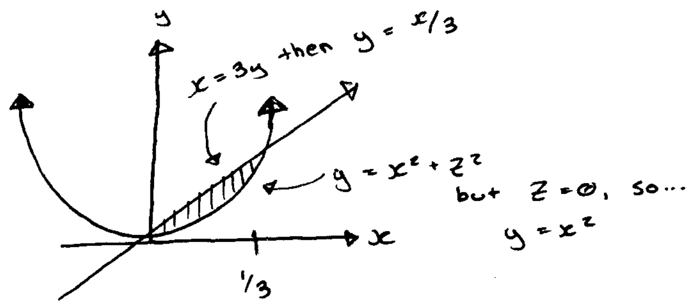
thus:

$$\text{Vol}(E) = \int_0^{\sqrt{2}} \int_{-\sqrt{8 - 4z^2}}^{\sqrt{8 - 4z^2}} \int_{2z}^{\sqrt{8 - y^2}} 1\, dx\, dy\, dz$$

(compute as a triple integral)

Ex. Find the volume of the solid (in the first octant) bounded by $y = x^2 + z^2$ and the planes $x = 3y$ and $z = 0$





$$E = \{(x, y, z) \quad 0 \le z \le \sqrt{y - x^2}$$
$$x^2 \le y \le x/3$$
$$0 \le x \le 1/3$$

$$\Rightarrow \int_0^{1/3} \int_{x^2}^{x/3} \int_0^{\sqrt{y - x^2}} 1 \, dz \, dy \, dx$$

$$\Rightarrow \int_0^{1/3} \int_{x^2}^{x/3} \sqrt{y - x^2} \, dy \Big) \, dx$$ hard to compute

trying ... (another solution)

$$E = \{(x, y, z) : x^2 + z^2 \le y \le x/3$$

(x, z) is D = domain in xz-plane

Intersection : $x^2 + z^2 = x/3$

$$z = \sqrt{x/3 - x^2}$$

then $0 \le z \le \sqrt{x/3 - x^2}$

$$0 \le x \le 1/3$$

$$\text{Volume}(E) = \iiint_E 1 \, dv = \int_0^{1/3} \int_0^{\sqrt{x/3 - x^2}} \int_{x^2}^{x/3} 1 \, dy \, dz \, dx$$