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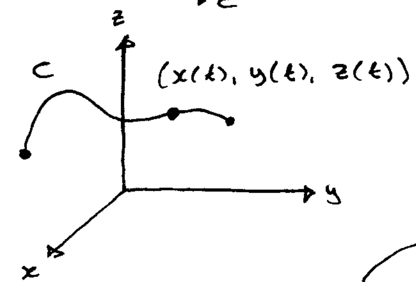
Line Integral for Scalar Function

$$\int_C f(\underbrace{x,y,z}_{C})\,ds = \int_a^b \underbrace{f(x(t),y(t),z(t))}_{\text{Function}} \cdot \underbrace{\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}}_{\text{arc length}}\,dt$$



$$C:\ \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \\ a \le t \le b \end{cases}$$

$$r(t) = x(t)\,i + y(t)\,j + z(t)\cdot k$$
$$= \langle x(t),\ y(t),\ z(t)\rangle$$
$$r'(t) = x'(t)\,i + y'(t)\,j + z'(t)\,k$$

Remark

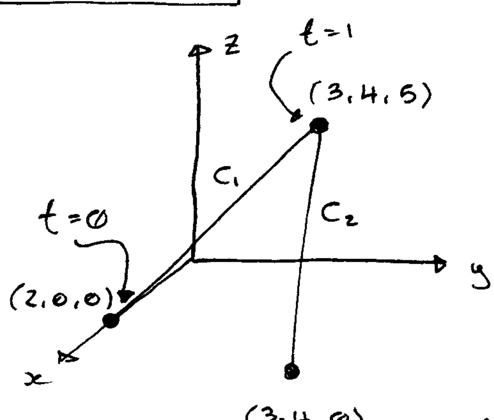
$$\int_C f(x,y,z)\,ds = \int_{-C} f(x,y,z)\,ds$$

$-C$: same curve traced backwards

$\boxed{\text{Example:}}$ Compute $\int_C (x+y+z)\,ds$

where C is the line segment C_1 from $(2,0,0)$ to $(3,4,5)$ followed by the vertical line segment C_2 from $(3,4,5)$ to $(3,4,0)$.

$\boxed{\text{Solution:}}$



$$\int_C (x+y+z)\,ds = \int_{C_1}(x+y+z)\,ds + \int_{C_2}(x+y+z)\,ds$$

C_1 < Point $(2,0,0)$, direction vector

$v = \langle 3-2,\ 4-0,\ 5-0\rangle$
$= \langle \underset{a}{1},\ \underset{b}{4},\ \underset{c}{5}\rangle$

$$x = 2+t = 2+t$$
$$y = 0+4t = 4t$$
$$z = 0+5t = 5t$$
$$0 \le t \le 1$$

$$r(t) = (2+t)\,i + 4t\,j + 5t\,k$$
$$r'(t) = 1\,i + 4\,j + 5\,k$$
$$\|r'(t)\| = \sqrt{1^2+4^2+5^2} = \sqrt{42}$$

Now: $\int_{C_1}(x+y+z)\,ds = \int_0^1 (\underbrace{2+t}_{x} + \underbrace{4t}_{y} + \underbrace{5t}_{z}) \cdot \underbrace{\sqrt{42}\,dt}_{\text{arc length}}$

$$= \sqrt{42}\int_0^1 (2+10t)\,dt = \sqrt{42}\left(2t + 10\frac{t^2}{2}\right)\Big|_{t=0}^{t=1} = 7\sqrt{42}$$

Some more versions of line integrals:

" $\int_C f(x,y,z)\,dx$ " $= \int_a^b (f(x(t)), f(y(t)), f(z(t)))\, x'(t)\,dt$ → line integral w.r.t. x

" $\int_C f(x,y,z)\,dy$ " $= \int_a^b (f(x(t)), f(y(t)), f(z(t)))\, y'(t)\,dt$ → line integral w.r.t. y

" $\int_C f(x,y,z)\,dz$ " $= \int_a^b (f(x(t)), f(y(t)), f(z(t)))\, z'(t)\,dt$ → line integral w.r.t. z

Notation

$$\int_{\underline{\underline{C}}} f(x,y,z)\,dx + \int_{\underline{\underline{C}}} g(x,y,z)\,dy + \int_{\underline{\underline{C}}} h(x,y,z)\,dz$$

→ " $\int_C f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)\,dz$ "

Example Evaluate $\int_C y^2\,dx + x\,dy$

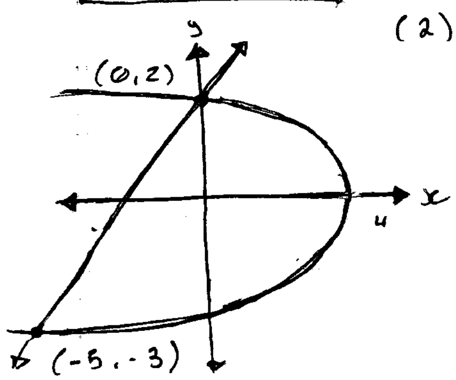
In two situations:

(1) $C = C_1$ the line segment from $(-5,-3)$ to $(0,2)$

(2) $C = C_2$ the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$

Solution (1) as in previous example

(2)



Remark: $\int_C f(x,y,z)\,dx = \int_a^b f(x(t), y(t), z(t))\,x'(t)\,dt$

$$\int_{\underline{\underline{C}}} f(x,y,z)\,dx = \underline{\underline{-}}\int_{\underline{\underline{-C}}} f(x,y,z)\,dx$$

C_2 : $x = 4 - t^2$

$y = t$

$-3 \le t \le 2$

CONT'D ...

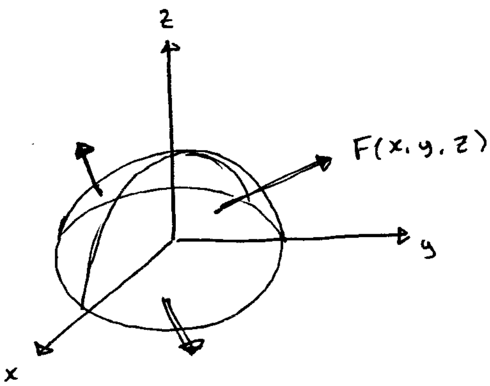
$$\int_{C_2} y^2dx + xdy = \int_{-3}^{2}\Big[\underbrace{t^2}_{y^2}\underbrace{(-2t)}_{x'(t)} + \underbrace{(4-t^2)}_{x}\underbrace{(1)}_{y'(t)}\Big]dt$$

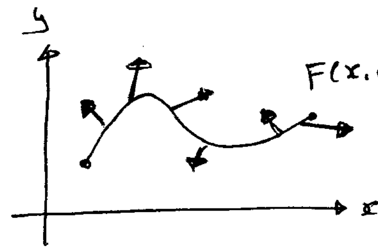
$$= \int_{-3}^{2}(-2t^3+4-t^2)dt = \cdots$$

Vector Fields

2-dimensional : $F(x,y)$ = 2-dimensional vector = $P(x,y)i + Q(x,y)j$

3-dimensional : $F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$





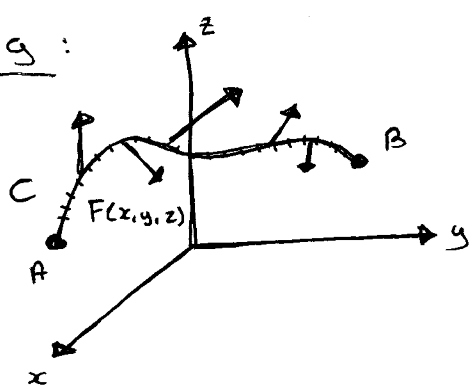
One or the other - they'll never mix.

Line Integral For Vector Fields

GOAL : To define " $\int_C F(x,y,z)$ "

C: curve in 3-dim

Setting :



How do we compute the total work done to move a particle along C under the action of a force field $F(x,y,z)$

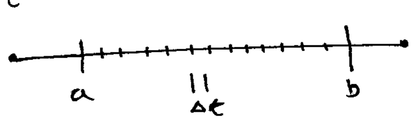
C : $x = x(t)$

$y = y(t)$

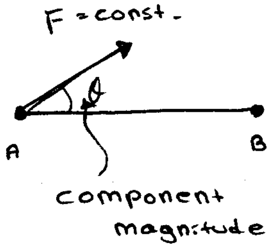
$z = z(t)$

$a \le t \le b$

t:

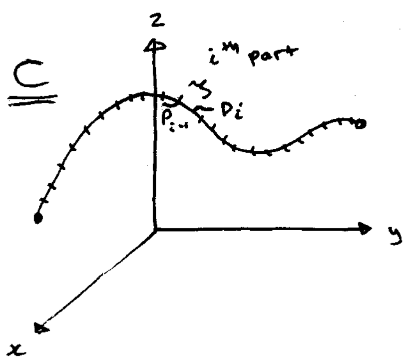


Special Case

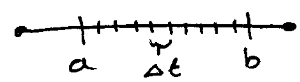


$$\text{Work} = \|F\| \cdot \cos\theta \; \|\overrightarrow{AB}\| = F \cdot \overrightarrow{AB}$$

$\|F\|\cos\theta$



t:



$$t_0 < t_1 < t_2 < \dots < t_n$$

Total work: $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n}$ work done on i^{th} part

$$= \lim_{n\to\infty} \sum_{i=1}^{n} F(x(t_i), y(t_i), z(t_i)) \cdot T(t_i) \underbrace{\Delta s}_{\text{arc length}}$$

$$= \int_C \underbrace{F \cdot T}_{\text{scalar}} \, ds = \int_a^b \underbrace{\left[F(x(t), y(t), z(t)) \cdot T(t)\right]}_{\text{dot product}} \cdot \underbrace{\|r'\| \, dt}_{\text{arc length part}}$$

$$= \int_a^b F(x(t), y(t), z(t)) \underset{\text{dot product}}{\cdot} \underbrace{\frac{r'(t)}{\|r'(t)\|}}_{T(t)}$$

$$= \int_a^b F(x(t), y(t), z(t)) \cdot r'(t) \, dt$$

Definition: Line Integrals of Vector Fields

$F(x, y, z)$ = 3-dimensional vector field

C = curve in 3-dim

$$\text{“}\int_C F \cdot dr\text{”} = \int_a^b F(x(t), y(t), z(t)) \cdot r'(t) \, dt$$

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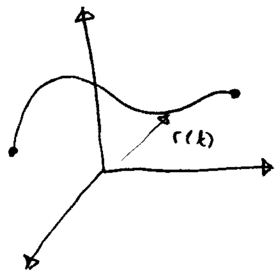
Given a curve C:



C: $x = x(t)$
$y = y(t)$
$z = z(t)$
$a \le t \le b$

$r(t) = x(t)\,i + y(t)\,j + z(t)\,k$

Two main types of line integrals:

(1) For scalar functions

$$\int_c f(x,y,z)\,ds = \int_a^b f(r(t))\cdot \underbrace{\|r'(t)\|\,dt}_{\text{arc length part}}$$

Note: $\int_c = \int_{-c}$

(2) For vector fields $F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$

$$\int_c F(x,y,z)\cdot dr = \int_a^b F(r(t))\cdot r'(t)\,dt$$

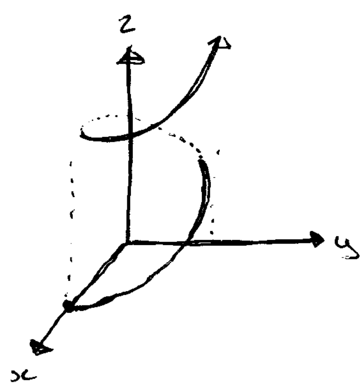
Note: $\int_c F(x,y,z)\cdot dr = -\int_{-c} F(x,y,z)\,dr$

Example: What is the total work done to move a particle along the helix.

$r(t) = (\cos t)i + (\sin t)j + t\,k$

with $0 \le t \le 2\pi$, under the action of the vector field

$F(x,y,z) = (-2x)i + (3y)j + (xy)k$



Solution

$r(t) = \cos(t)i + (\sin t)j + t\,k$

C: $x = \cos t$
$y = \sin t$
$z = t$

$0 \le t \le 2t$

$r'(t) = (-\sin t)i + (\cos t)j + k$

$$\int_C F(x,y,z)\cdot dr = \int_0^{2\pi} \Big[(\underbrace{-2\cos t}_{x})i + (\underbrace{3\sin t}_{y})j + (\underbrace{\cos t}_{x}\,\underbrace{\sin t}_{y}) \ldots$$

$$\cdot \big[(-\sin t)i + (\cos t)j + k \big] dt$$

$$\Rightarrow \int_0^{2\pi} (2\cos t\sin t) + (3\cos t\sin t) + (\cos t\sin t)\ dt$$

$$\Rightarrow \int_0^{2\pi} 6\cos t\ \sin t\ dt = 6\ \frac{(\sin^2 t)}{2}\Big|_{t=0}^{t=2\pi} = 0$$

Remark:

$$C: \boxed{\begin{aligned} x &= x(t) \\ y &= y(t) \\ z &= z(t) \\ a \le\ & t \le b \end{aligned}}$$

dot product

$$F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$$

$$\int_C F(x,y,z)\cdot dr = \int \Big[P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k \Big] \cdot \big[x'(t)i + y'(t)j + z'(t)k \big]$$

$$\to \int_a^b P(x(t),y(t),z(t))x'(t) + Q(x(t),y(t),z(t))y'(t) + R(x(t),y(t),z(t))z'(t)\,dt$$

$$\to \boxed{\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz}$$

Fundamental Theorem for Line Integrals

Input	Output	
$f(x,y,z)$ scalar Function	$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$	vector Fields

gradient of f

$\boxed{\text{Ex.}}$ Let $f(x,y) = x^2 - 3y^2 + 3$

Compute ∇f

Sol'n: $$\nabla f(x,y) = \underbrace{2x}_{\leftarrow \partial f/\partial x}\, i + \underbrace{(-6y)}_{\leftarrow \partial f/\partial y}\, j$$

$$= (2x)i - (6y)j$$

Ex: Let $F(x,y) = (2xy)i + (x^2-2y)j$

Find a scalar function $f(x,y)$ such that $F = \nabla f$

Sol: $F(x,y) = (2xy)i + (x^2-y)j$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

$$\frac{\partial f}{\partial x} = 2xy \rightsquigarrow f(x,y) = \int 2x\underset{\text{Fixed}}{y}\,dx = x^2y + \underbrace{g(y)}_{\text{constant}}$$

$$\frac{\partial f}{\partial y} = x^2 - y \rightsquigarrow f(x,y) = \int(\underset{\text{Fixed}}{x^2}-y)\,dy = x^2y - y^2/2 + \underbrace{h(x)}_{\text{constant}}$$

Take $g(y) = -y^2/2$

$h(x) = 0$

Answer

$$\boxed{f(x,y) = x^2y - y^2/2}$$

Question: Given a vector field

$F(x,y) = P(x,y)i + Q(x,y)j$

Can we always find a scalar function $f(x,y)$ such that $\mathbf{F} = \nabla f$?

Answer: $F(x,y) = P(x,y)i + Q(x,y)j$

$$\nabla f(x,y) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

Then we must have $\frac{\partial f}{\partial x} = P(x,y)$

$\partial f/\partial y = Q(x,y)$

$$\partial/\partial y(\partial f/\partial x) = \partial^2 f/\partial y\partial x = \partial^2 P(x,y)/\partial y$$

$$\partial/\partial x(\partial f/\partial y) = \partial^2 f/\partial x\partial y = \partial^2 Q(x,y)/\partial x$$

When f is "nice" one can prove $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$

This gives us $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$

<u>Theorem</u>: Given $F(x,y) = P(x,y)i + Q(x,y)j$

there exists a scalar function $f(x,y)$

satisfying $F = \nabla f$ if and only if

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

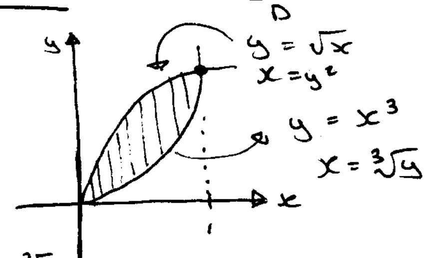
In such a case we can say that $F(x,y)$ is a conservative vector field.

#1 Change the order of integration

$$\int_0^1 \left(\int_{x^3}^{\sqrt{x}} f(x,y)\, dy \right) dx$$

From $dy\,dx$ to $dx\,dy$

Solution: $= \iint_D f(x,y)\, dA$



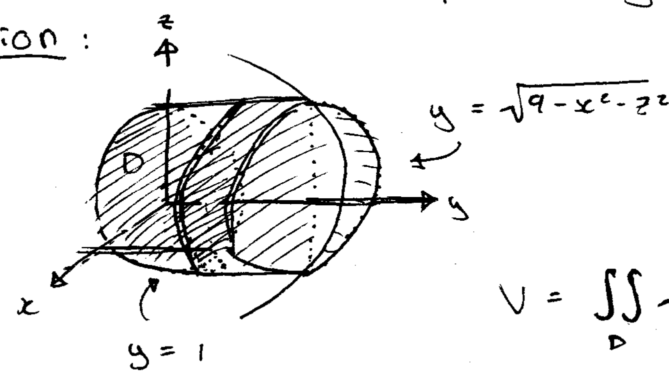
$$D = \left\{ (x,y) : \begin{array}{l} x^3 \le y \le \sqrt{x} \\ 0 \le x \le 1 \end{array} \right\}$$

$$= \left\{ (x,y) : \begin{array}{l} y^2 \le x \le \sqrt[3]{x} \\ 0 \le y \le 1 \end{array} \right\}$$

$$\int_0^1 \left(\int_{y^2}^{\sqrt[3]{y}} f(x,y)\, dx \right) dy$$

#2 Compute the volume of a solid bounded by the cylinder $4 = x^2 + z^2$, the plane $y = 1$ and the hemisphere $y = \sqrt{9 - x^2 - z^2}$

Solution:



$$V = \iint_D \sqrt{9-x^2-z^2}\, dA - \iint_D 1\, dA$$

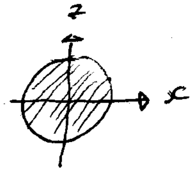
$$= \iint_D \underbrace{\sqrt{9-x^2-z^2}}_{\text{TOP}} - \underbrace{1\ dA}_{\text{BOTTOM}}$$

cylindrical $\longrightarrow$

$x = r\cos\theta$

$z = r\sin\theta$

$y = y$



$$\int_0^{2\pi} \int_0^2 \left(\sqrt{9-r^2} - 1 \right) \times r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left(r\sqrt{9-r^2} - r \right) dr\, d\theta$$

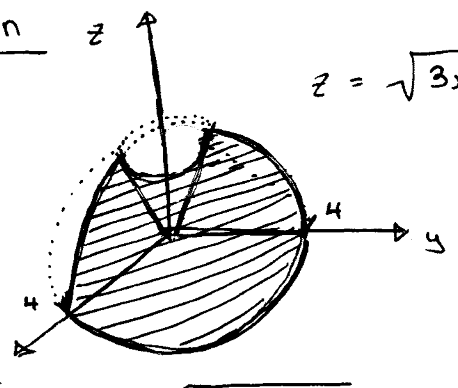
Substitute: $u = 9 - r^2$

$du = -2r\, dr$

$-\frac{1}{2} du = r\, dr$

(#3) Let E be the solid in the first octant bounded above by the cone $z = \sqrt{3x^2+3y^2}$ bounded below by the x-y plane and bounded on the side by the hemisphere $z = \sqrt{16-x^2-y^2}$. Find the volume.

Solution



$E = (\rho, \phi, \theta)$

$0 \le \theta \le \pi/2$

$\pi/6 \le \phi \le \pi/2$

$0 \le \rho \le 4$

Cone $z = \sqrt{3x^2+3y^2}$

$\rho\cos\phi = \sqrt{3(x^2+y^2)}$

$= \sqrt{3\rho^2\sin^2\phi}$

$\cancel{\rho}\cos\phi = \sqrt{3}\,\cancel{\rho}\sin\phi$

$1/\sqrt{3} = \tan\phi$

$\phi = \pi/6$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$> x^2+y^2 = \rho^2\sin^2\phi$

$z = \rho\cos\phi$

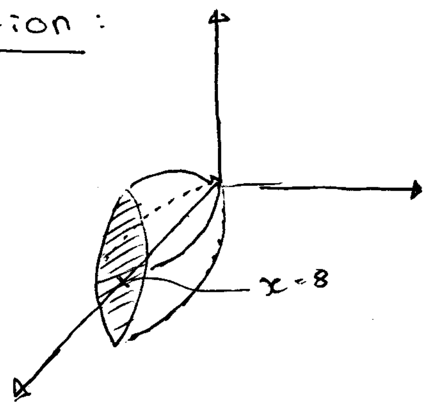
Volume of E $= \int_0^{\pi/2}\int_{\pi/6}^{\pi/2}\int_0^4 1\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

#4 Let C be the curve of intersection between paraboloid $x = 2y^2 + 2z^2$ and plane $x = 8$. Compute:

$$\int_C F \cdot dr$$

where $F(x, y, z) = yz\,i - z\,j + y\,k$ and C has direction of your choice.

Solution :



$$\begin{cases} x = 2y^2 + 2z^2 \\ x = 8 \end{cases}$$

$$8 = 2y^2 + 2z^2$$

$$4 = y^2 + z^2$$

$$C: \quad x = 8, \quad y = 2\cos t, \quad z = 2\sin t$$

$$0 \le t \le 2\pi$$

$$r(t) = (8i + 2\cos t\, j + 2\sin t\, k)$$

$$r'(t) = (0 + (-2\cos t)j + (2\cos t)k)$$

$$\int_C F \cdot dr = \int_0^{2\pi} [(2\cos t)(2\sin t)i - (2\sin t)j + (2\cos t)k] \cdot [(-2\sin t\, j + 2\cos t\, k] dt$$

$$\Rightarrow \int_0^{2\pi} (4\sin^2 t + 4\cos^2 t)\, dt$$

$$\Rightarrow \int_0^{2\pi} 4\, dt = 8\pi$$