- last time
  - Combinations, $\binom{n}{k} = \frac{n!}{(n-k!)k!}$
  - probability
  - equally likely outcomes $Pr(E) = \frac{N(E)}{N(S)}$

MARCH 5/18
APPLIED ANAL.

Axioms of Probability:

(i) $0 \le Pr(E) \le 1$, for all events E

(ii) $Pr(S) = 1$

(iii) if $E \cap F = \emptyset$, then $Pr(E \cup F) = Pr(E) + Pr(F)$

$E \cup \bar{E} = S$, $E \cap \bar{E} = \phi$

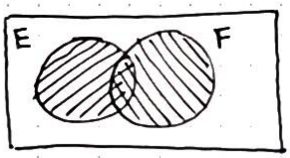
$Pr(E \cup \bar{E}) = Pr(E) + Pr(\bar{E})$

$Pr(S) = Pr(E) + Pr(\bar{E})$

$1 = Pr(E) + Pr(\bar{E})$

$Pr(\bar{E}) = (-Pr(E))$

$Pr(E)$ = sum of the probabilities of the outcomes in E.



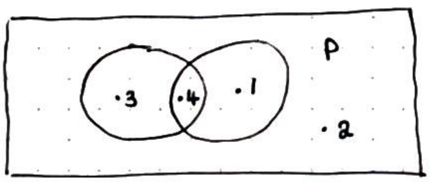
In general, $Pr(E \cup F) = Pr(E) + Pr(F) - Pr(E \cap F)$

- e.g. $Pr(E) = 5$, $Pr(F) = 6$, $Pr(E \cap F) = 8$, what is $Pr(E \cap F)$?

  $.8 = .5 + .6 - Pr(E \cap F)$, so $Pr(E \cap F) = .3$

We can put probability into the regions of Venn diagrams

- e.g. in a survey of 100 people, 70 liked cake, 50 liked pie, 40 liked both cake and pie. Find the probability that a randomly selected participant liked neither cake or pie.



.2

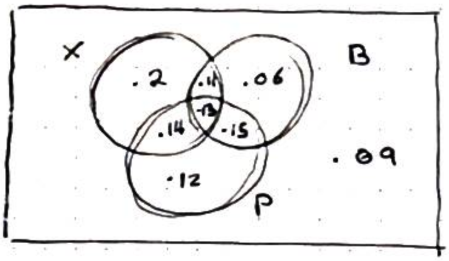
- e.g. in a survey of 100 people, 58 liked Xena, 46 liked Buffy, 54 liked South Park, 24 liked X + B, 27 liked X + P, 28 liked B + P, 13 liked all 3.

Find the probability that a randomly selected participant liked:

(i) Xena, nothing else $\rightarrow$ .2

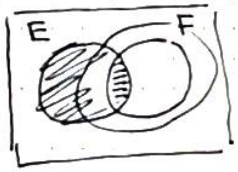
(ii) exactly one show $\rightarrow$ .38

(iii) Buffy + South Park $\longrightarrow$ .17 = 0.11 + 0.06



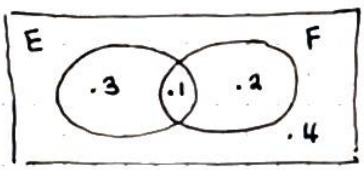
Let E and F be events, then the <u>conditional possibilities</u> of E given F is

$$\Pr(E|F) = \frac{\Pr(E \cap F)}{\Pr(F)}$$



- e.g. $\Pr(E) = .4$, $\Pr(F) = .3$, $\Pr(E \cap F) = .1$

Find (i) $\Pr(E|F)$ (ii) $\Pr(F|E)$ (iii) $\Pr(\bar{E}|F)$



(i) $\dfrac{\Pr(E \cap F)}{\Pr(F)} = \dfrac{(.1)}{(.3)} = \dfrac{1}{3}$ (ii) $\dfrac{\Pr(F \cap E)}{\Pr(E)} = \left.\dfrac{(0.1)}{(0.4)}\right) = \dfrac{1}{4}$

(iii) $= \dfrac{\Pr(\bar{E} \cap F)}{\Pr(F)} = \dfrac{(.2)}{(.3)} = \dfrac{2}{3}$

Events E and F are <u>independent</u> if $Pr(E \cap F) = Pr(E)Pr(F)$

If $Pr(F) \neq 0$, this means $\frac{Pr(E \cap F)}{Pr(F)} = Pr(E)$

so, $Pr(E|F) = Pr(E)$

if $Pr(E) = 0$, $\frac{Pr(F \cap E)}{Pr(E)} = Pr(E)$

so $Pr(F|E) = Pr(E)$

- e.g. Flip a balanced coin twice

Let E be the event that we get heads on Flip 1

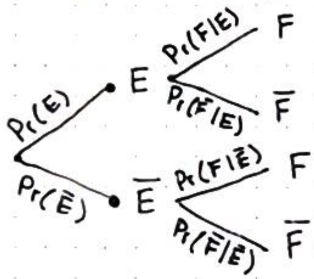
Let F " " Flip 2

$Pr(E) = Pr(F) = 1/2$

$\{HH, HT, TH, TT\}$ $Pr(E \cap F) = 1/4 = Pr(E)Pr F)$

E and F are independent

Note that $Pr(E)Pr(F|E) = Pr(E \cap F)$



$Pr(E \cap F) = Pr(E)Pr(F|E)$

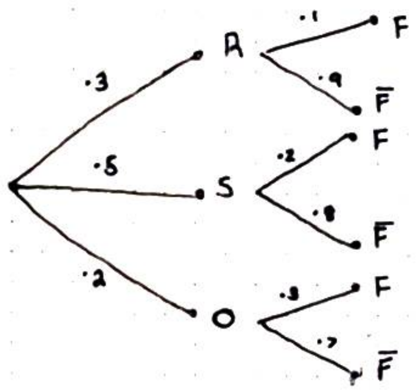
$Pr(E \cap \bar{F}) = Pr(E)Pr(\bar{F}|E)$

" ↪ "

" ↪ "

We can write conditional probability on the branches of a tree diagram, then multiply along the path to get the probability of the outcome.

- e.g. 30 % of the cars in the parking lot are red 50 % are silver, 20 % are some other colour. Further suppose 10 % of the red cars, 20 % of the silver cars, 30 % of the other cars have fuzzy dice.



$$\Pr(F) = (0.3)(0.1) + (0.5)(0.2) + \cdots$$
$$\cdots (0.2)(0.3) = 0.19$$

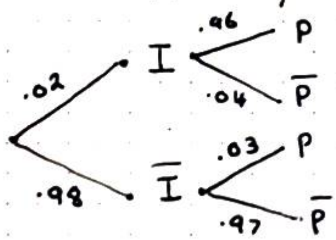
In <u>Bayes Probability</u>, we are given $\Pr(E|F)$, we want to know $\Pr(F|E)$. These can be solved with tree diagrams

- e.g. In the car problem, given that the car has fuzzy dice, find the prob. that it is red.

$$\Pr(R|F) = \frac{\Pr(R \cap P)}{\Pr(F)} = \frac{(0.3)(0.1)}{(0.19)} = \frac{3}{19}$$

- e.g. 2% of the population has Xena fever. There is a test, but it has a 3% false positive, and a 4% false neg. Given that a randomly selected person tests positive, what is the probability that he has Xena fever?

I = infected, P = tests positive



$$\Pr(I|P) = \frac{\Pr(I \cap P)}{\Pr(P)}$$

$$= \frac{(.02)(.96)}{(.02)(.96) + (.98)(.03)}$$

- e.g. Find the prob. of getting "one-pair" in poker

$$\rightarrow \frac{13\binom{4}{2}\binom{12}{3}4^3}{\binom{52}{5}}$$

- e.g. Find the prob. of getting at least 4 hearts, and exactly one 7

$$\rightarrow \frac{\binom{12}{4} + \binom{12}{3}36 + \binom{12}{4}3}{\binom{52}{5}}$$

Case 1 : All hearts

- e.g. Find the probability that our hand will contain two kings, and exactly one club

$$\rightarrow \frac{3\binom{36}{3} + \binom{3}{2}12\binom{36}{2}}{\binom{52}{5}}$$

Case 1 : Club is a king

Case 2 : Club isn't a king

- e.g. Fizzbin : You get 7 randomly dealt cards

A royal Fizzbin is three of a kind and two pairs

Find the prob of getting one : KKKQQ44

$$\rightarrow \frac{13\binom{4}{3}\binom{12}{2}\binom{4}{2}^2}{\binom{52}{7}}$$

- e.g. Our Xena Fan club has 23 members, including Bob. We must select a pres, tres. and vice pres. what are the odds of bob being president?

$$\frac{P(22,2)}{P(23,3)} = \frac{\cancel{22}\cdot\cancel{21}}{23\cdot\cancel{22}\cdot\cancel{21}} = \frac{1}{23}$$

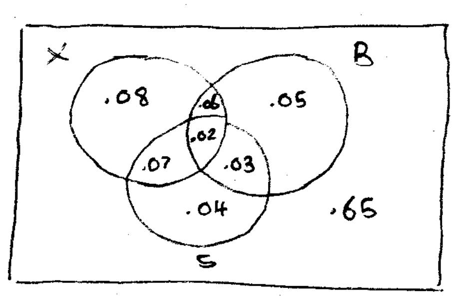
- e.g. We ask 100 people about their Favorite shows.

23 like Xena, 16 like Buffy, 16 like simpsons

8 like X + B, 9 like X + S, 5 like B + S

2 like all 3

(i) likes neither B or S



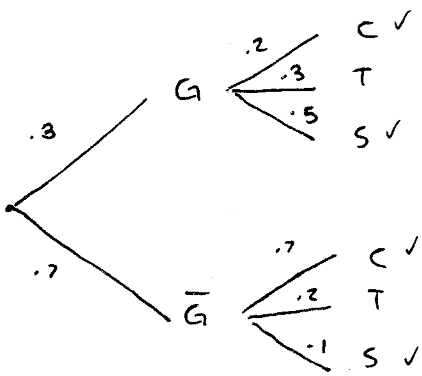
(i) $Pr(\bar{B} \cap \bar{S}) = 0.08 + 0.65$
$= 0.73$

(ii) $Pr(\bar{B} \mid X) = \frac{Pr(\bar{B} \cap X)}{Pr(X)}$

$\Rightarrow \frac{.08 \times .07}{.23} = \frac{15}{23}$

- 30% of the vehicles in the parking lot are green.
  - ↳ of the green vehicles, 20% cars, 30% trucks, 50% suv
  - ↳ of non green vehicles, 70% cars, 20% trucks, 10% suv

Given that a randomly selected vehicle is not a truck, find the prob. that it is green



$$\Pr(G \mid \bar{T}) : \frac{(.3)(.2) + (.3)(.5)}{(.3)(.2) + (.3)(.5) + (.7)(.7) + \ldots + (.7)(.1)}$$

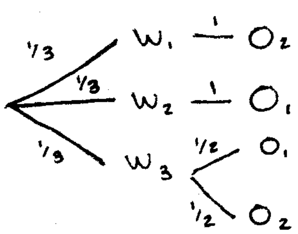
MARCH 10/18
APPLIED ANAL.

- Last time
  - axioms of probability
  - $Pr(\bar{E}) = 1 - Pr(E)$, $Pr(E \cup F) = Pr(E) + Pr(F) - Pr(E \cap F)$
  - $Pr(E) = \sum$ Prob of outcomes in E
  - Venn diagrams
  - Combined Prob. $Pr(E|F) = \dfrac{Pr(E \cap F)}{Pr(F)}$
  - independent
  - tree diagrams
  - Bayes' probabilities
- e.g. Monty-hall problem: You are shown three doors, behind one is a car, and the others goats. You choose door 3, Monty opens a door to reveal a goat. would you like to keep door 3, or switch for another door? Switch!

W_i = door i is the winner

O_i = monty opens door i

We chose door 3.



$$Pr(W_3|O_1) = \frac{Pr(W_3 \cap O_1)}{Pr(O_1)} = \frac{(1/3)(1/2)}{(1/3)(1)+(1/3)(1/2)} = 1/3$$
$\hookrightarrow = 1/3$

$$Pr(W_2|O_1) = \frac{Pr(W_2 \cap O_1)}{Pr(O_1)} = \frac{(1/3)(1)}{(1/3)(1)+(1/3)(1/2)} = 2/3$$
$\hookrightarrow = 2/3$

Chapter 4 - Probability Distributions

A <u>random variable</u> X assigns a numerical value to each possible outcome of an experiment

- e.g. roll 2 balanced dice, let x = total
- e.g. Flip a coin 3 times, let x be the number of heads

A <u>probability distribution</u> assigns a probability $f(x)$ to each possible value x of X

$f(x) = Pr(X = x)$

We usually denote this with a table

- e.g. Flip a balanced coin 3 times, Let x = # of heads

x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

- e.g. roll 2 balanced dice, let x be the total

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	1/36	2/36	3/36	4/36	5/36	6/36	[illegible]/36	4/36	3/36	2/36	1/36

- e.g. Flip a balanced coin until head appears

x	1	2	3	...	n	...
$f(x)$	1/2	1/4	1/8	...	$\frac{1}{2^n}$	...

To get a valid distribution we must have:

$f(x) \geq 0$ for all x

$$\sum_{\text{all } x} f(x) = 1$$

- e.g.

x	-762	18	2394
$f(x)$	1/2	1/6	1/3

Valid

<u>Cumulative distribution</u> = $F(x) = Pr(X \leq x)$

$$\Rightarrow \sum_{y \leq x} f(y)$$

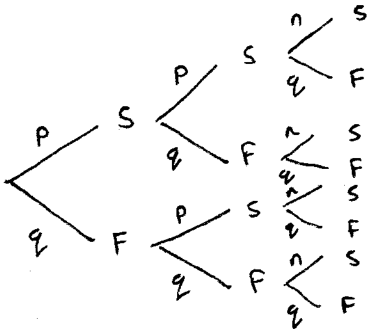
- e.g.

x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8
$F(x)$	1/8	1/2	3/8	1

A <u>Bernouli process</u> (or <u>Bernouli trials</u>) is a series of trials, with the following assumptions:

1) At each stage, there are two possibilities, success (S) and Failure (F)
2) The probability of success, P, is the sum of each stage
3) The stage is independent
4) The number of stages, n, is predetermined

We are interested in the number of success, x



# of success	prob.
S 3	p^3
F 2	$p^2 \varepsilon$
S 2	$p^2 \varepsilon$
F 1	$p \varepsilon^2$
S 2	$p^2 \varepsilon$
F 1	$p \varepsilon^2$
S 1	$p \varepsilon^2$
F 0	ε^3

In general each prob. leading to x-successes in n-attempts

has Probability $p^x q^{n-x} = p^x (1-p)^{n-x}$

There are $\binom{n}{x}$