- last time - invertible matrix.

  - inverse $AA^{-1} = A^{-1}A = I_n$.

  - $(A^{-1})^{-1} = A$, $(AB)^{-1} = B^{-1}A^{-1}$; $(A^T)^{-1} = (A^{-1})^T$.

  - $(A|I_n) \to (I_n|A^{-1})$

    $\searrow$ (zero row $|$?), A not invertible.

  - adjoint matrix.

  - A invertible $\Leftrightarrow \det(A) \neq 0$.

    In this case, $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

  - $AX = B$, A invertible $\Rightarrow X = A^{-1}B$.

  - homogeneous system - trivial/nontrivial solution.

  - A square, then $AX = 0$ has only the trivial solution

    $\Leftrightarrow A$ invertible.

If we have a system of equations $AX = B$ and A is invertible,

we can use <u>Cramer's rule</u>:

For each i, let A_i be the matrix obtained by replacing column i of A with B.

Then $x_i = \dfrac{\det(A_i)}{\det(A)}$.

e.g. solve $\begin{array}{l} x_1 + 2x_2 = 5 \\ 3x_1 + 9x_2 = 7. \end{array}$ $A = \begin{pmatrix} 1 & 2 \\ 3 & 9 \end{pmatrix}$, $\det(A) = 3$.

$A_1 = \begin{pmatrix} 5 & 2 \\ 7 & 9 \end{pmatrix}$, $\det(A_1) = 31$.

$A_2 = \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix}$, $\det(A_2) = -8$.

$x_1 = \dfrac{31}{3}$, $x_2 = \dfrac{-8}{3}$.

e.g. solve $\begin{array}{l} x_1 \quad\;\; + x_3 = 1 \\ x_1 + 2x_2 + x_3 = 5 \\ x_1 + 2x_2 + 3x_3 = 7. \end{array}$ $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

$\det(A) = 1(-1)^{1+1}\det\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} + 1(-1)^{1+3}\det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

$= 4 + 0 = 4.$

$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 5 & 2 & 1 \\ 7 & 2 & 3 \end{pmatrix}$ $\det(A_1) = 1(-1)^{1+1}\det\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} + 1(-1)^{1+3}\det\begin{pmatrix} 5 & 2 \\ 7 & 2 \end{pmatrix}$

$= 4 + (-4) = 0.$

$A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 7 & 3 \end{pmatrix}$. $\det(A_2) = 1(-1)^{1+1}\det\begin{pmatrix} 5 & 1 \\ 7 & 3 \end{pmatrix} + 1(-1)^{1+2}\det\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} + 1(-1)^{1+3}\det\begin{pmatrix} 1 & 5 \\ 1 & 7 \end{pmatrix}$

$= 8 - (2) + 2 = 8.$

$A_3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 5 \\ 1 & 2 & 7 \end{pmatrix}$, $\det(A_3) = 1(-1)^{1+1}\det\begin{pmatrix} 2 & 5 \\ 2 & 7 \end{pmatrix} + 1(-1)^{1+3}\det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

$= 4 + 0 = 4.$

$x_1 = \frac{0}{4} = 0,\ x_2 = \frac{8}{4} = 2,\ x_3 = \frac{4}{4} = 1.$

Let A be an $n\times n$ matrix. A vector in $\mathbb{R}^n$ will be regarded as an $n\times 1$ column vector.

We say that a number λ is an eigenvalue for A if there is a nonzero vector $x \in \mathbb{R}^n$ s.t. $Ax = \lambda x$.

We say that x is an eigenvector corresponding to λ.

If we take all the eigenvectors corresponding to λ, including the zero vector, we obtain the eigenspace corresponding to λ.

Checking if x is an eigenvector of A is easy: calculate Ax and see if it is a scalar multiple of x.

e.g. $A = \begin{pmatrix}1&2\\2&1\end{pmatrix}$. Are these eigenvectors? (i) $\begin{pmatrix}1\\1\end{pmatrix}$ (ii) $\begin{pmatrix}-1\\1\end{pmatrix}$ (iii) $\begin{pmatrix}3\\4\end{pmatrix}$.

(i) $\begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix} = 3\begin{pmatrix}1\\1\end{pmatrix}$, so it is an eigenvector corresponding to $\lambda = 3$.

(ii) $\begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} = -1\begin{pmatrix}-1\\1\end{pmatrix}$, " " " " " " " " $\lambda = -1$.

(iii) $\begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}11\\10\end{pmatrix}$, $\begin{pmatrix}3\\4\end{pmatrix}$ is not an eigenvector.

If λ is an eigenvalue of A, then its eigenspace is a subspace of $\mathbb{R}^n$.

Indeed: $A(0)=0 \overset{=\lambda 0}{}$, so 0 is in the eigenspace.

Let $Ax=\lambda x$, $Ay=\lambda y$. $A(x+y)=Ax+Ay=\lambda x+\lambda y$
$=\lambda(x+y)$.

$\therefore x+y$ is in the eigenspace.

Let $Ax=\lambda x$, $\mu\in\mathbb{R}$. Then $A(\mu x)=\mu Ax=\mu\lambda x=\lambda(\mu x)$

$\therefore \mu x$ is in the eigenspace.

If we have an eigenvalue for A, we would like to describe its eigenspace. We can find a basis for the eigenspace.

e.g. $A=\begin{pmatrix}1&1\\1&1\end{pmatrix}$. Its eigenvalues are 0 and 2.

$\lambda=2$: solve $Ax=2x$, i.e. $Ax-2x=0$.

$Ax-2Ix=0$. $(A-2I)x=0$.

$A-2I=\begin{pmatrix}1&1\\1&1\end{pmatrix}-2\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}-1&1\\1&-1\end{pmatrix}$.

$\left(\begin{array}{cc|c}-1&1&0\\1&-1&0\end{array}\right) \xrightarrow[\text{①}\leftrightarrow\text{②}]{\text{swap}} \left(\begin{array}{cc|c}1&-1&0\\-1&1&0\end{array}\right) \xrightarrow[\text{to ②}]{\text{add ①}} \left(\begin{array}{cc|c}1&-1&0\\0&0&0\end{array}\right)$.

Let $x_2 = t$
$x_1 - t = 0 \Rightarrow x_1 = t$
$x_2 = t$.

The eigenspace is $\left\{\begin{pmatrix} t \\ t \end{pmatrix} : t \in \mathbb{R}\right\}$.

To find a basis for the eigenspace, take each parameter in turn: let it equal 1, all others = 0.

Basis: $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.

$\lambda = 0$: $(A - 0I)x = 0$.

$A - 0I = A$, $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow[\text{to } (2)]{\text{add } -(1)} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

Let $x_2 = t$, $x_1 = -t$. The eigenspace is $\left\{\begin{pmatrix} -t \\ t \end{pmatrix} : t \in \mathbb{R}\right\}$

Basis: $\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$.

If A has eigenvalue λ, to find the eigenvectors, solve $(A - \lambda I)x = 0$. To find a basis for each eigenspace, let each parameter in turn be 1, and all others 0.

e.g. let $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Its eigenvalues are $-1, 3$.

$\lambda = -1$: $A - (-1)I = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

$\left(\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \xrightarrow[\text{by } 1/2]{\text{mult } (1)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \xrightarrow[\text{to } (2)]{\text{add } -2(1)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \xrightarrow[(2)(3)]{\text{swap}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{by } 1/4]{\text{mult } (2)}$

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. Let $x_2 = t$, $x_1 = -t$, $x_3 = 0$.

Eigenspace: $\left\{\begin{pmatrix} -t \\ t \\ 0 \end{pmatrix} : t \in \mathbb{R}\right\}$. Basis: $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$

$\lambda = 3$: $A - 3I = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{by } -\frac{1}{2}]{\text{mult } (1)} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{to } (2)]{\text{add } -2(1)} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Let $x_2 = t$, $x_3 = u$, then $x_1 = t$.

Eigenspace: $\left\{\begin{pmatrix} t \\ t \\ u \end{pmatrix} : t, u \in \mathbb{R}\right\}$. Basis: $\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$.

- last time — Cramer's rule.

- eigenvalues, eigenvectors, eigenspaces.

$Ax = \lambda x$.

- given x, calculate Ax, see if it is a scalar multiple of x.

- given λ: solve $(A-\lambda I)x = 0$

- find a basis for eigenspace.

λ is an eigenvalue for A if and only if

$(A-\lambda I)x = 0$ has a nontrivial solution.

This happens if and only if $A-\lambda I$ is not invertible.

But this ~~can~~ occurs if and only if $\det(A-\lambda I) = 0$.

We call $\det(A-\lambda I)$ the <u>characteristic polynomial</u> of A.

We call $\det(A-\lambda I) = 0$ the <u>characteristic equation</u>.

To find eigenvalues, we solve the characteristic equation.

We then know how to find eigenvectors

- e.g. $A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$. Find eigenvalues, eigenvectors.

$$0 = \det(A - \lambda I) = \det\left(\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\begin{pmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2 - 25 = \lambda^2 - 2\lambda - 24$$

$$= (\lambda - 6)(\lambda + 4), \text{ so } \lambda = 6, -4.$$

$\lambda = 6$: $A - 6I = \begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix}$.

$$\left(\begin{array}{cc|c} -5 & 5 & 0 \\ 5 & -5 & 0 \end{array}\right) \xrightarrow[\text{by } -1/5]{\text{mult } ①} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 5 & -5 & 0 \end{array}\right) \xrightarrow[\text{to } ②]{\text{add } -5①} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

let $x_2 = t$
$x_1 = t$. Basis for eigenspace: $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.

$\lambda = -4$: $A - (-4)I = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 5 & 5 & 0 \\ 5 & 5 & 0 \end{array}\right) \xrightarrow[\text{by } 1/5]{\text{mult } ①} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 5 & 5 & 0 \end{array}\right) \xrightarrow[\text{to } ②]{\text{add } -5①} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

let $x_2 = t$
$x_1 = -t$. Basis: $\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$.

- e.g. $A = \begin{pmatrix} 1 & 4 \\ -3 & 1 \end{pmatrix}$, $0 = \det\begin{pmatrix} 1-\lambda & 4 \\ -3 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 12 \geq 12$.
No real eigenvalues.

- e.g. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

$$0 = \det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda)\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} - \det\begin{pmatrix} 1 & 1 \\ 1 & 1-\lambda \end{pmatrix} + \det\begin{pmatrix} 1 & 1-\lambda \\ 1 & 1 \end{pmatrix}$$

$$= (1-\lambda)\left((1-\lambda)^2 - 1\right) - (-\lambda) + (\lambda)$$

$$= (1-\lambda)(\lambda^2 - 2\lambda) + 2\lambda$$

$= -\lambda^3 + 3\lambda^2 = \lambda^2(-\lambda+3)$.

$\lambda = 0, 3$.

$\lambda = 0$: $A - 0I = A$

$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ \boxed{1} & 1 & 1 & 0 \\ \boxed{1} & 1 & 1 & 0 \end{array}\right) \begin{matrix} \text{add } -1 (1) \text{ to } (2) \\ \text{add } -1 (1) \text{ to } (3) \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

Let $x_2 = t$, $x_3 = u$. Then $x_1 = -t-u$. Basis $= \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$\lambda = 3$: $A - 3I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$.

$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \begin{matrix} \text{swap} \\ (1), (2) \end{matrix} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ \boxed{-2} & 1 & 1 & 0 \\ \boxed{1} & 1 & -2 & 0 \end{array}\right) \begin{matrix} \text{add } 2(1) \text{ to } (2) \\ \text{add } -(1) \text{ to } (3) \end{matrix} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \begin{matrix} \text{mult } (2) \\ \text{by } -\frac{1}{3} \end{matrix}$

$\left(\begin{array}{ccc|c} 1 & \boxed{-2} & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & \boxed{3} & -3 & 0 \end{array}\right) \begin{matrix} \text{add } 2(2) \text{ to } (1) \\ \text{add } -3(2) \text{ to } (3) \end{matrix} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. Let $x_3 = t$. $x_1 = t$, $x_2 = t$. Basis $= \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

If A is upper/lower triangular, the eigenvalues are the diagonal entries

- e.g. $A = \begin{pmatrix} 3 & 4 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 7 \end{pmatrix}$. $0 = \det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 4 & 5 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 7-\lambda \end{pmatrix}$

$= (3-\lambda)(2-\lambda)(7-\lambda)$.

Let A be a square matrix, n a positive integer. Then

$A^n = \underbrace{AA \cdots A}_{n \text{ times}}$.

- e.g. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$.

$A^m A^n = A^{m+n}$

$(A^m)^n = A^{mn}$.

If we have a polynomial $f(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n$

then $f(A) = c_0 I + c_1 A + c_2 A^2 + \dots + c_n A^n$.

CAYLEY-HAMILTON THM: A satisfies its characteristic equation.

e.g. $A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$, char poly: $\lambda^2 - 2\lambda - 24$.

$\longrightarrow$ $A^2 - 2A - 24I = 0$.

$$\begin{pmatrix} 26 & 10 \\ 10 & 26 \end{pmatrix} - 2\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix} - 24\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We can use this to calculate higher powers of matrices.

If A is $n \times n$ then its characteristic poly has degree n.

We can write A^n as a linear combination of $I, A, A^2, \dots, A^{n-1}$.

(e.g. above $A^2 = 2A + 24I$).

This allows us to write all powers of A in terms of

$I, A, A^2, \dots, A^{n-1}$.

(e.g. also $A^2 = 2A + 24I$

$$A^3 = A^2A = 2A^2 + 24A$$

$$= 2(2A + 24I) + 24A$$

$$= 28A + 48I.)$$

If $A^m = c_0 I + c_1 A + c_2 A^2 + \cdots + c_{n-1} A^{n-1}$

then for every eigenvalue λ of A

$\lambda^m = c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_{n-1}\lambda^{n-1}$.

If we have n different eigenvalues, we can obtain a system of n equations in n unknowns, and solve for $c_0, c_1, \ldots, c_{n-1}$.

- last time - characteristic polynomial $\det(A-\lambda I)$
  - " equation $\det(A-\lambda I)=0$
  - eigenvalues satisfy this equation
  - A triangular - eigenvalues are diagonal entries.
  - A^m, polynomials in terms of A
  - Cayley-Hamilton Thm.
  - if A is $n\times n$, $A^m = c_0 I + c_1 A + c_2 A^2 + \cdots + c_{n-1} A^{n-1}$.
  - the eigenvalues satisfy the same equation.

- e.g. $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. Find A^{10}.

As A is 2×2, $\underline{A^{10} = c_0 I + c_1 A}$.

As A is upper triangular, the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2$.

$$c_0 + c_1 \lambda = \lambda^{10}.$$

$$c_0 + c_1 = 1$$
$$c_0 + 2c_1 = 1024$$
$$c_1 = 1023, \quad c_0 = -1022.$$

$$A^{10} = -1022\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 1023\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

ex. $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, find A^6.

~~[illegible]~~ $c_0 I + c_1 A = A^6$.

$0 = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$

$= (\lambda - 3)(\lambda + 1)$.

$\lambda = 3, -1$.

$c_0 + c_1 \lambda = \lambda^6$

$$\begin{aligned} c_0 + 3c_1 &= 729 \\ c_0 - c_1 &= 1 \\ \hline 4c_1 &= 728, \quad c_1 = 182, \quad c_0 = 183. \end{aligned}$$

$A^6 = 183 I + 182 A$.

A matrix A is <u>symmetric</u> if $A = A^T$.

ex. $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & -7 \end{pmatrix}$.

THM: If A is a symmetric matrix, all of its eigenvalues are real!

Thm: If A is symmetric, and it has two different eigenvalues λ and μ, then if $Ax = \lambda x$ and $Ay = \mu y$, then x and y are orthogonal.

PF: $y^TAx = y^T\lambda x = \lambda y^Tx = \lambda(x\cdot y)$. $(3\ 5\ 7)\begin{pmatrix}1\\2\\-6\end{pmatrix}$

$\parallel$

$y^TAx = y^TA^Tx = (Ay)^Tx = (\mu y)^Tx = \mu(x\cdot y)$.

$\lambda(x\cdot y) = \mu(x\cdot y)$. So $x\cdot y = 0$ (yay) or $\lambda = \mu$ $\cancel{\text{⨳}}$.

e.g. $A = \begin{pmatrix}1&2&0\\2&1&0\\0&0&3\end{pmatrix}$. Let $x = \begin{pmatrix}1\\1\\1\end{pmatrix}$. $Ax = \begin{pmatrix}3\\3\\3\end{pmatrix} = 3x$.

$y = \begin{pmatrix}-1\\1\\0\end{pmatrix}$. $Ay = \begin{pmatrix}-1\\1\\0\end{pmatrix} = (-1)y$.

We see that $x\cdot y = 0$.

An $n\times n$ matrix A is <u>orthogonal</u> if it is invertible, and $A^{-1} = A^T$.

e.g. $A = \begin{pmatrix}1/\sqrt{2}&-1/\sqrt{2}&0\\1/\sqrt{2}&1/\sqrt{2}&0\\0&0&1\end{pmatrix}$, $AA^T = \begin{pmatrix}1/\sqrt{2}&-1/\sqrt{2}&0\\1/\sqrt{2}&1/\sqrt{2}&0\\0&0&1\end{pmatrix}\begin{pmatrix}1/\sqrt{2}&1/\sqrt{2}&0\\-1/\sqrt{2}&1/\sqrt{2}&0\\0&0&1\end{pmatrix}$

$= \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

THM: An $n\times n$ matrix A is orthogonal if and only if its columns form an orthonormal set

Suppose $A = (x_1\ x_2 \cdots x_n)$. Then

$A^TA = \begin{pmatrix}x_1^T\\x_2^T\\\vdots\\x_n^T\end{pmatrix}(x_1 \cdots x_n) = \begin{pmatrix}x_1^Tx_1 & x_1^Tx_2 & \cdots\\ x_2^Tx_1 & x_2^Tx_2 & \cdots\end{pmatrix}$.

For this to be I, $\underset{\underset{x_i\cdot x_j}{\parallel}}{x_i^Tx_j} = \begin{cases}1, & i=j\\0, & i\neq j\end{cases}$.

Let A be a square matrix. Then A is said to be diagonalizable if there is an invertible matrix P so that

$P^{-1}AP = D$, for some diagonal matrix D.

Thm: If A is $n \times n$, then A is diagonalizable $\iff$ it has n linearly independent eigenvectors.

Suppose we have $Ax_i = \lambda_i x_i$. Let $P = (x_1 x_2 \cdots x_n)$.

$AP = (Ax_1 \, Ax_2 \cdots Ax_n) = (\lambda_1 x_1 \, \lambda_2 x_2 \cdots \lambda_n x_n)$.

Let $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$.

$PD = (x_1 \cdots x_n)\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = (\lambda_1 x_1 \cdots \lambda_n x_n)$.

$AP = PD$. Now, P is invertible if and only if its columns are linearly independent.

Given an $n \times n$ matrix A, we find the eigenvalues, and a basis for each eigenspace. If n vectors were obtained, A is diagonalizable.

Let P = the matrix with columns equal to these basis vectors.

Then $P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ where the λ_i are the eigenvalues in the order in which they were used.

If we get fewer than n basis vectors, A is not diagonalizable.

COROLLARY: If A is $n \times n$ and has n different eigenvalues, it is diagonalizable.

- e.g. $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. $0 = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4$

$= \lambda^2 - 2\lambda - 3$

$\lambda = 3, -1$. $\qquad = (\lambda - 3)(\lambda + 1)$

$\lambda = 3$: $A - 3I = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$.

$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right) \xrightarrow[\text{by } -1/2]{\text{mult } ①} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right) \xrightarrow[\text{to } ②]{\text{add } -2①} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$.

Let $x_2 = t$, $x_1 = t$. Basis: $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.

$\lambda = -1$: $A - (-1)I = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$.

$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right) \xrightarrow[\text{by } 1/2]{\text{mult } ①} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right) \xrightarrow[\text{to } ②]{\text{add } -2①} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

Let $x_2 = t$, $x_1 = -t$. Basis: $\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$.

$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $P^{-1}AP = D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$.