Examples of Vector spaces: JAN. 22/18

APPLIED ANAL.

$\mathbb{R}$?

P - the set of polynomials

$(x^3+2x+1)+(-x^3+x^2) = x^2+2x+1$

P_n - the set of polynomials of degree at most n (including the zero polynomial)

Let V be a vector space. The subset W of V is a <u>subspace</u> if it is a vector space using the same addition and scalar multiplication.

- e.g. P_n is a subspace of P
- e.g. P_3 is a subspace of P_5
- e.g. $\{\emptyset\}$ and V are subspaces of V

THM: Let W be a subset of a vector space V. Then W is a subspace of V if and only if

(i) $0 \in W$

(ii) if $w_1, w_2 \in W$ then $w_1 + w_2 \in W$ (closure under addition)

(iii) if $w \in E$, $\lambda \in \mathbb{R}$, then $\lambda \in \mathbb{R}$, then $\lambda w \in W$ (closure under scalar mult.)

- e.g. let $V = \mathbb{R}^3$, $W = \{\langle a, b, 2a+3b\rangle : a, b \in \mathbb{R}\}$ is W a subspace of V?

(i) let $a = b = \emptyset$. Then $\langle \emptyset, \emptyset, \emptyset\rangle \in W$

(ii) Take $\langle a_1, b_1, 2a_1 + 3b_1\rangle, \langle a_2, b_2, 2a_2 + 3b_2\rangle \in W$

$\langle a_1 + a_2, b_1 + b_2, 2(a_1+a_2) + 3(b_1+b_2)\rangle \in W$

(iii) Take $\langle a, b, 2a+3b\rangle \in W$, $\lambda \in \mathbb{R}$

$\lambda\langle a, b, 2a+3b\rangle = \langle \lambda a, \lambda b, 2\lambda a + 3\lambda b\rangle \in W$

It is a subspace

- e.g. let $V = \mathbb{R}^2$, $W = \{\langle a, a^2\rangle : a \in \mathbb{R}\}$

(i) $\langle \emptyset, \emptyset\rangle \in W$

(ii) Take $\langle a_1, a_1^2\rangle, \langle a_2, a_2^2\rangle \in W$

$\langle a_1, a_2^2\rangle + \langle a_2, a_2^2\rangle = \langle a_1 + a_2, a_1^2 + a_2^2\rangle$

As $a_1^2 + a_2^2 \neq (a_1+a_2)^2$, this will not be in W in general

$\langle 1, 1\rangle \in W$, $\langle 2, 4\rangle \in W$ $\quad \langle 1, 1\rangle + \langle 2, 4\rangle = \langle 3, 5\rangle \in W$

W is <u>not</u> closed under addition, so not a subspace

Let V be a vector space, $v_1, \dots, v_k \in V$

A vector $v \in V$ is a <u>linear combination</u> of $v_1, \dots, v_k$ if $v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_k v_k$ for some $\lambda_1, \dots, \lambda_k \in \mathbb{R}$

- e.g. in $\mathbb{R}^2$, $3\langle 1,2\rangle - 4\langle 2,3\rangle = \langle -5,-6\rangle$

  $\langle -5,-6\rangle$ is a linear combination of $\langle 1,2\rangle$ and $\langle 2,3\rangle$

- e.g. Is $\langle 1,2,3\rangle$ a linear combination of $\langle -1,0,1\rangle$ and $\langle 2,0,6\rangle$?

  No! $\lambda_1\langle -1,0,1\rangle + \lambda_2\langle 2,0,6\rangle = \langle -\lambda_1+2\lambda_2, 0, \lambda_1+6\lambda_2\rangle \neq \langle 1,2,3\rangle$

Let V be a vector space, $v_1,\dots,v_k \in V$. These vectors are <u>linearly dependent</u> if there exist $\lambda_1,\dots\lambda_k \in \mathbb{R}$, not all zero, so that $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_k v_k = 0$.
If not they are <u>linearly independent.</u>

- e.g. $\vec{i}, \vec{j}, \vec{k}$ in $\mathbb{R}^3$ are linearly independent

  Suppose $\lambda_1\vec{i} + \lambda_2\vec{j} + \lambda_3\vec{k} = \vec{0}$

  $\langle \lambda_1,\lambda_2,\lambda_3\rangle = \langle 0,0,0\rangle$. so $\lambda_1 = \lambda_2 = \lambda_3 = 0$

- e.g. in P, the vectors $1, x, x^2, x^3$ are linearly dependent

  Suppose $\lambda_1(1) + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3$ is the zero polynomial

  Then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$

- e.g. are $\langle 1,0,2\rangle, \langle 3,0,0\rangle, \langle 2,-1,8\rangle$ linearly independent in $\mathbb{R}^3$?

  Suppose $\lambda_1\langle 1,0,2\rangle + \lambda_2\langle 3,0,0\rangle + \lambda_3\langle 2,-1,8\rangle = \langle 0,0,0\rangle$

  $\cancel{\lambda_1} + 2\lambda_1 + 3\cancel{\lambda_2} + \cancel{2\lambda_3} - \lambda_3 + 8\lambda_3 = \langle 0,0,0\rangle$

  $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$. They are linearly independent.

- e.g. in $\mathbb{R}^2$, $\langle 1,2\rangle, \langle 2,3\rangle, \langle -5,-6\rangle$ are linearly dep.

  $3\langle 1,2\rangle - 4\langle 2,3\rangle = \langle -5,-6\rangle$

  $3\langle 1,2\rangle - 4\langle 2,3\rangle - 1\langle -5,-6\rangle = \langle 0,0\rangle$

Let V be a vector space, $v_1,\dots,v_k = V$ such that $v_1,\dots,v_k$ <u>span</u> V if every vector in V is a linear combination of $v_1,\dots,v_k$.

- e.g. $\vec{i}, \vec{j}, \vec{k}$ span $\mathbb{R}^3$

  $\langle a_1,a_2,a_3\rangle = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

- e.g. $1, x, x^2, x^3, \dots, x^n$ span P_n

  $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n = a_0 1 + a_1 x + a_2 x^2 + \dots + a_n x^n$

- e.g. there is no finite set of vectors that spans P

If $v_1, \ldots, v_k \in V$, then the span of $v_1, \ldots, v_k$ is the set of all linear combinations of $v_1, \ldots, v_k$. It is a subspace of V

(i) $0 = 0v_1 + 0v_2 + \ldots + 0v_k$ is in the span

(ii) $(\lambda_1 v_1 + \ldots + \lambda_k v_k) + (\mu_1 v_1 + \ldots + \mu_k v_k)$

$= (\lambda_1 + \mu_1)v_1 + \ldots + (\lambda_k + \mu_k)v_k$ is in the span.

(iii) $\mu(\lambda_1 v_1 + \ldots + \lambda_k v_k) = \mu\lambda_1 v_1 + \ldots + \mu\lambda_k v_k$ is in the span

Let V be a vector space, then $v_1, \ldots, v_k \in V$ are said to be a basis for V if they are linearly independent and span V.

- e.g. $\vec{i}, \vec{j}, \vec{k}$ form a basis for $\mathbb{R}^3$
- e.g. in $\mathbb{R}^n$, let $e_1 = \langle 1, 0, 0, 0, \ldots, 0 \rangle$, $e_2 = \langle 0, 1, 0, 0, \ldots, 0 \rangle$ $\ldots$ $e_n = \langle 0, 0, 0, 0, \ldots, 1 \rangle$.

Then $e_1, \ldots, e_n$ form the standard basis for $\mathbb{R}^n$.

- e.g. in P_n, $1, x, x^2, x^3, \ldots, x^n$ is the standard basis
- e.g. P has no simple basis

If V is a vector space, then the number of vectors is a basis for V is fixed, and is called the dimension of V. $\dim V$. If V has no simple basis it is infinite-dimensional.

- e.g. $\dim \mathbb{R}^n = n$
- e.g. $\dim P_n = n+1$
- e.g. P is infinite dimensional
- e.g. $W = \{\langle a, b, 2a+3b \rangle = a, b \in \mathbb{R}\}$

$\langle a, b, 2a+3b \rangle = a\langle 1, 0, 2 \rangle + b\langle 0, 1, 3 \rangle$

$\langle 1, 0, 2 \rangle$, $\langle 0, 1, 3 \rangle$ span W

They are linearly independent and $\therefore \dim W = 2$

In any vector space V, v_1, v_2 and linearly independent provided neither is a scalar multiple of the other.

[ONLY use with vectors]

Suppose $\lambda_1 v_1 + \lambda_2 v_2 = 0$. If $\lambda_1 = 0$,

$\lambda_1 v_1 = -\lambda_2 v_2$, so $v_1 = \frac{-\lambda_2}{\lambda_1} v_2$

If $\lambda_2 \neq 0$, $\lambda_2 v_2 = -\lambda_1 v_1$

$v_2 = \frac{-\lambda_1}{\lambda_2} v_1$

Last time:

- Subspace, Subspace test
- linear combination
- linear dependence / independence
- span
- basis
- dimension

JAN. 24/18
Applied Anal.

Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$. Then $\vec{v}_1, \dots, \vec{v}_k$ are said to be an orthogonal set of vectors is $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$

- e.g. $\langle 1, 1, 0 \rangle$, $\langle 1, -1, 0 \rangle$, $\langle 0, 0, 3 \rangle$

hense that $\vec{v}_1, \dots \vec{v}_k \in \mathbb{R}^n$ are an orthogonal set of vectors if they are an orthogonal set and each is a unit vector.

We can normalize an orthogonal set of nonzero vectors to obtain an orthogonal set.

- e.g. above example $(1/\sqrt{2}) \langle 1, 1, 0 \rangle$, $((1/\sqrt{2}) \langle 1, -1, 0 \rangle$ $(1/3) \langle 0, 0, 3 \rangle$ is an orthogonal set.

Orthogonal means $\vec{v_i} \cdot \vec{v_j} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0 & i \neq j \end{cases}$
"Kronecker delta"

We know that if V is a subspace of $\mathbb{R}^n$ and $\vec{v}_1, \dots, \vec{v}_k$ is a basis for V, then for any $\vec{v} \in V$ there exist $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ so that $\vec{v} = \lambda_1 \vec{v}_1 + \dots + \lambda_k \vec{v}_k$ but finding $\lambda_1, \dots, \lambda_k$ can be a lot of work.

But if $\vec{v}_1, \dots, \vec{v}_k$ is an orthanormal basis, then

$$\vec{v} = (\vec{v} \cdot \vec{v}_1)\vec{v}_1 + (\vec{v} \cdot \vec{v}_2)\vec{v}_2 + \dots + (\vec{v} \cdot \vec{v}_k)\vec{v}_k$$

Suppose $\vec{v} = \lambda_1 \vec{v}_1 + \dots + \lambda_k \vec{v}_k$

Then $\vec{v} \cdot \vec{v}_i = \lambda_1 \vec{v}_1 \cdot \vec{v}_i + \lambda_2 \vec{v}_2 \cdot \vec{v}_i + \dots + \lambda_k \vec{v}_k \cdot \vec{v}_i = \lambda_i$

- e.g. $\vec{v}_1 = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle$ $\vec{v}_2 = \langle 1/\sqrt{2}, -1/\sqrt{2}, 0 \rangle$
$\vec{v}_3 = \langle 0, 0, 1 \rangle$

write $\langle 3, 5, 2 \rangle$ as a linear combination

$\langle 3, 5, 2 \rangle \cdot \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle = 8/\sqrt{2}$

$\langle 3, 5, 2 \rangle \cdot \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle = -2/\sqrt{2}$

$\langle 3, 5, 2 \rangle \cdot \langle 0, 0, 1 \rangle = 2$

$\langle 3, 5, 2 \rangle \cdot \frac{8}{\sqrt{2}} \vec{v}_1 - \frac{2}{\sqrt{2}} \vec{v}_2 + 2\vec{v}_3$

② ... Suppose we have a basis $\vec{U_1}, \dots, \vec{U_k}$ for a V or $\mathbb{R}$?

We use the Gram-Schmidt algorithm to obtain an orthonormal basis. We will obtain an orthanormal basis $\vec{v_1}, \dots, \vec{v_k}$ for V, then normailize we obtain an orthonormal basis $\vec{w_1}, \dots, \vec{w_k}$.

$\vec{V_1} = \vec{u_1}$

$\vec{V_2} = \vec{u_2} - \frac{\vec{u_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \cdot \vec{v_1} - \frac{\vec{u_3} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} \cdot \vec{v_2}$

$\vdots$

$\vec{V_k} = \vec{u_k} - \frac{\vec{u_k} \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \cdot \vec{v_1} - \frac{\vec{u_k} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} \vec{v_2} - \dots - \frac{\vec{u_k} \vec{v_{k-1}}}{\vec{v_{k-1}} \cdot \vec{v_{k-1}}} \vec{v_{k-1}}$

At any stage, we can multiply $\vec{v_i}$ by a non-zero scalar.

- e.g. V is a subspace of $\mathbb{R}^3$ with basis $\vec{u_1} = \langle 1, 3, 2 \rangle$ $u_2 = \langle 3, 2, 0 \rangle$. Find orthonormal basis for V.

$\vec{V_1} = \vec{u_1} = \langle 1, 3, 2 \rangle$

$\vec{V_2} = \vec{u_2} - \frac{\vec{u_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} = \langle 3, 2, 0 \rangle - (9/14) \langle 1, 3, 2 \rangle$

Replace $\vec{V_2}$ with $14 \vec{V_2}$

$\vec{V_2} = 14 \langle 3, 2, 0 \rangle - 9 \langle 1, 3, 2 \rangle = \langle 33, 1, -18 \rangle$

Normalize: $\vec{w_1} = 1/\sqrt{14} \langle 1, 3, 2 \rangle$, $\vec{w_2} = \frac{1}{\sqrt{1089 + 1 + 324}} \langle 33, 1, -18 \rangle$

- e.g. Let V be a subspace of $\mathbb{R}^4$ with the series $\vec{u_1} = \langle 1, 1, 0, 0 \rangle$, $\vec{u_2} = \langle 2, -1, 0, 0 \rangle$, $\vec{u_3} = \langle 1, 1, -1, 0 \rangle$

Find an orthonormal basis

$\vec{v_1} = \vec{u_1} = \langle 1, 1, 0, 0 \rangle$

$\vec{v_1} = \vec{u_2} - \frac{\vec{u_2} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \vec{v_1} = \langle 2, -1, 0, 0 \rangle - (1/2) \langle 1, 1, 0, 0 \rangle$

Replace $\vec{V_2}$ with $2\vec{V_2}$

$\vec{V_2} = 2 \langle 2, -1, 0, 0 \rangle - 1 \langle 1, 1, 0, 0 \rangle = \langle 3, -3, 0, 0 \rangle$

$\vec{V_3} = \vec{u_3} - \frac{\vec{u_3} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \cdot \vec{v_1} - \frac{\vec{u_3} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} \vec{v_2} = \langle 1, 1, -1, 0 \rangle - (2/2) \langle 1, 1, 0, 0 \rangle \dots$

$\dots (0/18) \langle 3, -3, 0, 0 \rangle$

$= \langle 0, 0, -1, 0 \rangle$

Normalize $\vec{w_1} = 1/\sqrt{2} \langle 1, 1, 0, 0 \rangle$ $\vec{w_2} = (1/\sqrt{18}) \langle 3, -3, 0, 0 \rangle$

$\vec{w_3} = \langle 0, 0, -1, 0 \rangle$

Chapter 8 - Matrices (8.1 - 8.10, 8.12, 8.14, 8.15)

Let m and n be positive integers. Then an $m \times n$ <u>matrix</u> is a rectangular array of numbers with m rows, n columns

- e.g. 2×3 matrix: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 12 & 0 \end{pmatrix}$ 3×2 matrix $\begin{pmatrix} -2 & 7 \\ 6 & 5 \\ 18 & 7 \end{pmatrix}$

We use capital letters to denote matrices.
If A is out matrix, then we write a_{ij} for the (ij)-entry that is, the entry in row i, column j

- e.g. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $a_{23} = 6$, $a_{31} = 7$

The <u>diagonal entries</u> are a_{ii}

- e.g. above, 1, 5, 9

A <u>square matrix</u> has the same number of rows as columns
A <u>column vector</u> is a matrix with 1 column
A <u>row vector</u> is a matrix with 1 row

- e.g. Column vector: $\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$, row vector $(-2, 1, 0, 4, 7)$

The $m \times n$ <u>zero matrix</u> has zeroes for all entries.
Denote the matrix O

We write $A = B$ if A and B are both $m \times n$ matrices and $a_{ij} = b_{ij}$ for all i, j

- e.g. $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \neq \begin{pmatrix} 5 & 3 \\ 1 & 2 \end{pmatrix}$

Let A and B be $m \times n$ matrices then their <u>sum</u> $A + B$ is the $m \times n$ matrix C so that $C_{ij} = a_{ij} + b_{ij}$ for all ij

- e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 7 & 2 & -1 \end{pmatrix} + \begin{pmatrix} -3 & 4 & 5 \\ 2 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1+(-3) & 2+4 & 3+5 \\ 7+2 & 0+2 & -1+4 \end{pmatrix} = \begin{pmatrix} -2 & 6 & 8 \\ 9 & 2 & 3 \end{pmatrix}$

If A and B do not have the same size their sum is undefined.

If A is an $m \times n$ matrix, its <u>negative</u> $-A$ is found by taking the negative of each entry.

- e.g. $-\begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 6 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & -1 & 2 \\ -3 & -2 & -6 \end{pmatrix}$

Properties of Matrix addition:

Let A, B, C be $m \times n$ matrices. Then

(i) $A+B$ is an $m \times n$ matrix

(ii) $(A+B)+C = A+(B+C)$ (associativity)

(iii) $A+B = B+A$ (commutivity)

(iv) $A+O = A$ (additive)

(v) $A+(-A) = O$ (additive merge)

→ Find all 3-Forms of the line through $(1,3,5)$ and $(2,-1,7)$

Vector: $(x, y, z) \cdot (1,3,5) + t(1,-4,2)$

Parametric: $x = 1+t$, $y = 3-4t$, $z = 5+2t$

Symmetric: $\frac{x-1}{1} = \frac{y-3}{(-4)} = \frac{z-5}{2}$

→ A triangle has verticies $(1,2,0)$, $(2,1,-1)$, $(4,3,7)$

Find the area.

Side vectors: $\vec{a} = \langle 1,-1,-1 \rangle$

$\vec{b} = \langle 2,2,8 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -1 \\ 2 & 2 & 8 \end{vmatrix} \Rightarrow \begin{vmatrix} -1 & -1 \\ 2 & 8 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ 2 & 8 \end{vmatrix} j + \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} k$$

$$\Rightarrow \langle -6, -10, 4 \rangle$$

$$\text{Area} = \tfrac{1}{2}\sqrt{(-6^2)+(-10)^2+(4)^2} = \tfrac{1}{2}\sqrt{150} \text{ ish}$$

↳ cross products only exist in R^3 space.

(the same question with

→ Let V be the set of all polynomials in the Form

$a + bx + (a+b)x^2$, $a, b \in \mathbb{R}$

Is V a subspace of P?

Let $a = b = 0$: $0 \leftarrow V$

$(a_1 + b_1x + (a_1+b_1)x^2) + (a_2 + b_2x + (a_2+b_2)x^2)$

$= (a_1 + a_2) + (b_1+b_2)x + (a_1+b_1+a_2+b_2)x^2$

$= (a_1 + a_2) + (b_1+b_2)x + ((a_1+a_2) + (b_1+b_2))x^2 \in V$

$\lambda(a + bx + (a b)x^2)$

$= \lambda a + \lambda b x + (\lambda a + \lambda b)x^2 \in V$

V is a subspace

→ e.g. Let $V = \{ \langle a, b, |a+b| \rangle , a,b \in \mathbb{R} \}$

is V a subspace of $\mathbb{R}^3$

Let $a = b = 0 = \langle 0,0,0 \rangle \in V$

$\langle a_1, b_1, |a_1 + b_1| \rangle + \langle a_2, b_2, |a_2 + b_2| \rangle$

$= \langle a_1 + a_2, b_1 + b_2, |a_1 + b_1| + |a_2 + b_2| \rangle$

$\langle 1,1,2 \rangle + \langle -1,-1,2 \rangle = \langle 0,0,4 \rangle \notin V$

Not a subspace, not closed under addition

→ Are these polynomials linearly dependent or independent?

$3 + 4x^2, \; 2 - 7x + 2x^2, \; 6x^2$

$\lambda_1(3 + 4x^2) + \lambda_2(2 - 7x + 2x^2) + \lambda_3(6x^2) = 0 + 0x + 0x^2$

$(3\lambda_1 + 2\lambda_2) - (7\lambda_2)x + (4\lambda_1 + 2\lambda_2 + 6\lambda_3)x^2 = 0 + 0x + 0x^2$

$3\lambda_1 + 2\cancel{\lambda_2} = 0$ / $-7\lambda_2 = 0$ / $4\cancel{\lambda_1} + 2\cancel{\lambda_2} + 6\cancel{\lambda_3} = 0$

$\lambda_1 = 0$ / $\lambda_2 = 0$ / $\lambda_3 = 0$

∴ linearly <u>independent</u>

→ Let V be the subspace of $\mathbb{R}^3$ with basis

$\vec{u}_1 = \langle 1,1,1 \rangle, \; \vec{u}_2 = \langle 1,-3,1 \rangle, \; \vec{u}_3 = \langle 1,0,2 \rangle$

Find an orthogonal set for V

Graham-Schmidt : $\vec{v}_1 = \vec{u}_1 = \langle 1,1,1 \rangle$

$\vec{v}_2 = \vec{u}_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} = \langle 1,-3,1 \rangle - (-\tfrac{1}{3}) \langle 1,1,1 \rangle$

Mult by 3: $\vec{v}_2 = 3\langle 1,-3,1 \rangle + \langle 1,1,1 \rangle = \langle 4,-8,4 \rangle$

Mult by $\tfrac{1}{4}$: $\vec{v}_2 = \langle 1,-2,1 \rangle$

$\vec{v}_3 = \vec{u}_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 v_2}{v_2 v_2} v_2$

$\Rightarrow \langle 1,0,2 \rangle - (\tfrac{2}{3})\langle 1,1,1 \rangle - (\tfrac{3}{6})(\tfrac{1}{2})\langle 1,-2,1 \rangle$

mult by 2 : $\vec{v}_3 = 2\langle 1,0,2 \rangle - 2\langle 1,1,1 \rangle - \langle 1,-2,1 \rangle$

$= \langle -1,0,1 \rangle$

Normalize : $(\tfrac{1}{\sqrt{3}})\langle 1,1,1 \rangle, (\tfrac{1}{\sqrt{6}})\langle 1,-2,1 \rangle, (\tfrac{1}{\sqrt{2}})\langle -1,0,1 \rangle$

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APPLIED ANAL.

- Last time: orthogonal / orthonormal set
  - orthonormal set
  - Graham - Schmitt
  - $m \times n$ matrix
  - Square matrix ($m = n$)
    → row vector, column vector
  - zero vector
  - Matrix addition

Let A be an $m \times n$ matrix, $\lambda \in \mathbb{R}$, then the scalar multiple λA is the $m \times n$ matrix B so that $b_{ij} = \lambda a_{ij}$ for all i, j

e.g. $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 0 & 6 \end{pmatrix}$, $\lambda A = \begin{pmatrix} 4 & 8 \\ 12 & -4 \\ 0 & 24 \end{pmatrix}$

Properties: let A and B be $m \times n$ matrices $\lambda, \mu \in \mathbb{R}$

(i) λA is an $m \times n$ matrix (closure under scalar mult.)

(ii) $\lambda(A+B) = \lambda A + \lambda B$ (distributive law)

(iii) $(\lambda + \mu)A = \lambda A + \mu A$ (" ")

(iv) $\lambda(\mu A) = (\lambda\mu)A$

(v) $1A = A$

The $m \times n$ matrices form a vector space

Let $A = (a_{11}, a_{12}, \dots, a_{1n})$ be a row vector $(1 \times n)$

Let $B = \begin{pmatrix} b_{11} \\ b_{21} \\ \dots \\ b_{n1} \end{pmatrix}$ be a column vector $(n \times 1)$

Then $AB = (a_{11}b_{11} + a_{12}b_{12} + \dots + a_{1n}b_{n1})$ (a 1×1 matrix)

- e.g. $(1, 3, 2)\begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} = ((1)(4) + (3)(5) + (2)(7)) = 33$

Let A be an $m \times n$ matrix and B be an $m \times n$ matrix

Then the product AB is the $m \times n$ matrix C so that

$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$

- e.g. $\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 4 & -3 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} (1)(2)+(3)(4) & (1)(1)+(3)(-3) & (1)(0)+(3)(2) \\ (2)(1)+(5)(4) & (2)(1)+(5)(-3) & (2)(0)+(5)(2) \end{pmatrix}$

2×2 2×3 $\rightarrow \begin{pmatrix} 14 & -8 & 6 \\ 24 & -13 & 10 \end{pmatrix}$

2×3

- e.g. $\begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 10 & 21 \end{pmatrix}$

2×3 3×2 2×2

Properties: (i) $(AB)C = A(BC)$ - (associvity)
(ii) $A(B+C) = AB + AC$ - (distributive law)
(iii) $(B+C)A = BA + CA$ - (" ")

Matrix multiplication is <u>not</u> Commutative!

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$AB = \emptyset \neq A = \emptyset$ or $B = \emptyset$

The $m \times n$ <u>identity matrix</u> I_n has all of the diagonal entries as "1", everything else is $\emptyset$

- e.g. $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

If A is an $m \times n$ matrix then $I_m A = A = A I_n$

Let A be an $m \times n$ matrix, then its' transpose A^T is the $m \times n$ matrix B so that $b_{ij} = a_{ji}$ for all i, j

- e.g. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

Properties: Let A, B be $m \times n$ matrices, C is $m \times$

$\lambda \in \mathbb{R}$

(i) $(A+B)^T = A^T + B^T$

(ii) $(\lambda A)^T = \lambda(A^T)$

(iii) $(A^T)^T = A$

(iv) $(AC)^T = C^T A^T$

Let A be a square matrix. Then A is the <u>upper triangular</u> matrix if all the entries below are zero.

- e.g. $\begin{pmatrix} 1 & 4 & -2 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

A is lower diagonal matrix is all entries above are zero.

- e.g $\begin{pmatrix} 3 & 0 \\ 1 & 4 \end{pmatrix}$

In a diagonal matrix, all entries off the diagonal are zero

- e.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

A <u>system of linear equations</u> has the form

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

This is a system of m equations in n unknowns
The x_i are the <u>variables</u>, the a_{ij} are the <u>coefficients</u> and the b_i are the <u>constants</u>

- e.g. $2x_1 + 5x_2 = 6$ 2 eqs, 2 unknowns
$-7x_1 + 2x_2 = 14$

A <u>solution</u> is a system is an n-tuple $(c_1, \dots, c_n)$ so that letting $x_1 = c_1, x_2 = c_2, \dots, x_n = c_n$, all equations are satisfied simultaneously.

To <u>solve</u> a system means to find all possible solutions

The system is <u>inconsistent</u> if there are no solutions
Otherwise, it is <u>consistent</u>, and there may be one solution or infinitely many.

The following <u>elementary row operations</u> can be performed without changing the solution to the system.

1) Multiply an equation by a non-zero constant
2) Add a multiple of one equation to another
3) Swap row equations

- e.g. solve
$$2x_1 + 6x_2 = 8 \quad (1)$$
$$4x_1 + 11x_2 = 1 \quad (2)$$

Multiply (1) by $(\frac{1}{2})$
$$x_1 + 3x_2 = 4 \quad (1)$$
$$4x_1 + 11x_2 = 1 \quad (2)$$

Add -4(1) to (2)
$$x_1 + 3x_2 = 4 \quad (1)$$
$$-x_2 = -15 \quad (2)$$

Multiply (2) by -1
$$x_1 + 3x_2 = 4 \quad (1)$$
$$x_2 = 15 \quad (2)$$

Add -3(2) to (1)
$$x_1 = -41 \quad (1)$$
$$x_2 = 15 \quad (2)$$

$x_1 = -41$, $x_2 = 15$ (unique solution)

If our system of equations is:
$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

... (3)

(3) ... then the <u>coefficient matrix</u> is

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

And the <u>augmented matrix</u> is

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} & b_m \end{array}\right)$$

The augmented matrix conveys all the information about the system. we can perform <u>elementary row operations</u> that do not change the solution to the system.

1) Multiply a row by a nonzero constant
2) Add a multiple of one row to another
3) swap two rows

- e.g. $2x_1 + 6x_2 = 8$ Form the augmented matrix:
  $4x_1 + 11x_2 = 1$

$$\left(\begin{array}{cc|c} 2 & 6 & 8 \\ 4 & 11 & 1 \end{array}\right) \xrightarrow{\text{Mult. } (1) \text{ by } 1/2} \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 4 & 11 & 1 \end{array}\right) \xrightarrow{\text{add } -4 (1) \text{ to } (2)} \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -1 & -15 \end{array}\right)$$

$$\xrightarrow{\text{mult } (2) \text{ by } -1} \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & 15 \end{array}\right) \xrightarrow{\text{add } -3 (2) \text{ to } (1)} \left(\begin{array}{cc|c} 1 & 0 & -41 \\ 0 & 1 & 15 \end{array}\right).$$

$\therefore \quad x_1 = -41$
$\quad\;\; x_2 = 15$