JAN. 15/17

Applied Anal.

Cross Product

Another operation with vectors, but this time the resut is a vector.

$$\left.\begin{array}{l} a = a_1 i + a_2 j + a_3 k \\ b = b_1 i + b_2 j + b_3 k \end{array}\right\} \quad a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Side: 3×3 determinant

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \cdots$$

$$\cdots - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

then

$$a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$$= (a_2 b_3 - a_3 b_2) i - (a_1 b_3 - a_3 b_1) j + (a_1 b_2 - a_2 b_1$$

Proposition: Given two vectors a and b:

(1) $a \times b$ is perpendicular to <u>both</u> a and b

(2) $a \times b = - b \times a$

(3) $\|a \times b\| = \|a\| \cdot \|b\| \cdot \sin\theta$ (where θ is the angle between them)

Proof: (1) We want to check that $(a \times b) \cdot a = 0$

(and thus $a \times b$ is perpendicular to a)

$$(a \times b) \cdot a = (a_2 b_3 - a_3 b_2) a_1 - (a_1 b_3 - a_3 b_1) a_2 - \cdots$$
$$\cdots (a_1 b_2 - a_2 b_1) a_3$$

$$\Rightarrow \cancel{a_1 a_2 b_3} - \cancel{a_1 a_3 b_2} - \cancel{a_1 a_2 b_3} + \cancel{a_2 a_3 b_1} + \cancel{a_1 a_3 b_2} \cdots$$
$$\cdots - \cancel{a_2 a_3 b_1} = \boxed{0} \quad \blacksquare$$

(3) We can check that

$$\|a \times b\|^2 = \|a\|^2 \cdot \|b\|^2 \cdot \sin^2\theta$$

First: $\|a \times b\|^2 = (a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2$

$$\left[x = x_1 i + x_2 j + x_3 k \;:\; \|x\| = (x \cdot x)^{1/2} = \sqrt{x_1^2 + x_2^2 + x_3^2} \right]$$

Second: $\|a\|^2 \cdot \|b\|^2 (\sin^2\theta) = \|a\|^2 \cdot \|b\|^2 (1 - \cos^2\theta)$

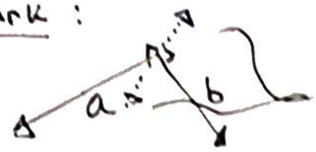
$$= \|a\|^2 \cdot \|b\|^2 - \|a\|^2 \cdot \|b\|^2 (1 - \cos^2\theta)$$

$$= \|a\|^2 \cdot \|b\|^2 - (a \cdot b)^2$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= \dots \blacksquare$$

Remark:



two potential cross-products

↳ Direction of $a \times b$ satisfies RHR.

Remark: (1) $i \times j$

$i = 1i + 0j + 0k$

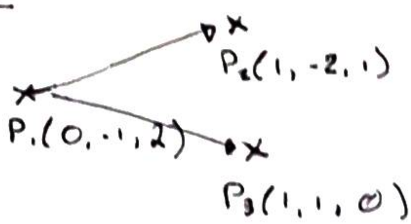
$j = 0i + 1j + 0k$

$$i \times j = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} i - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} j + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} k = 1k \checkmark$$

Example: Find a vector perpendicular to the plane determined by $P_1(0,-1,2)$, $P_2(1,-2,1)$ and $P_3(1,1,0)$

Solution:



$n = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$

$\overrightarrow{P_1P_2} = (1-0)i + [-2-(-1)]j + \dots$

$\dots (1-2)k = i - j - k$

$\overrightarrow{P_1P_3} = (1-0)i + [1-(-1)]j + \dots$

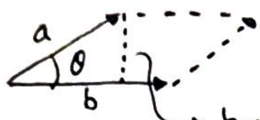
$\dots (0-2)k = i + 2j - 2k$

$$n = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ 1 & 2 & -2 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} -1 & -1 \\ 2 & -2 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} j + \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} k \Rightarrow 4i + j + 3k$$

Other geometric applications of cross products

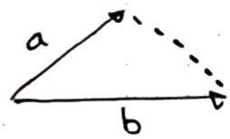
(1) Remember: $\|a \times b\| = \underbrace{\|a\| \cdot \|b\| \cdot \sin\theta}_{}$ → area of parallelogram formed by a and b



→ height $= \|a\| \cdot \sin\theta$

length of base $= \|b\|$

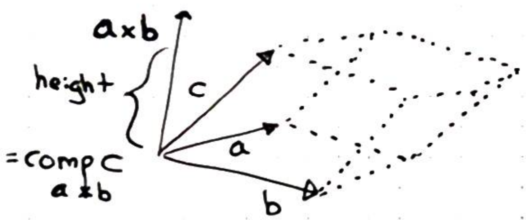
(2)



Area of the triangle formed by vectors a and b

$= \frac{1}{2}\|a \times b\|$

(3) Volume of the parallelipiped determined by 3 vectors $\underline{\underline{a}}$, $\underline{\underline{b}}$ and $\underline{\underline{c}}$



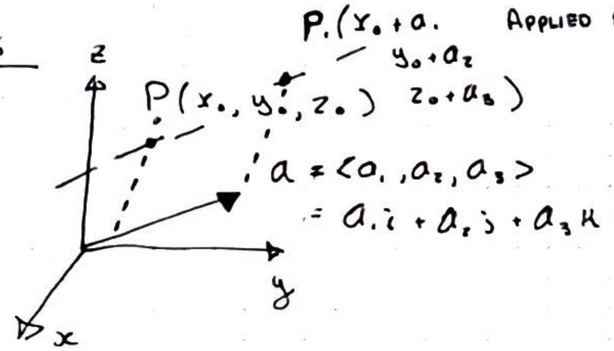
$V = (\text{area of base}) \times \text{height}$

$= \underbrace{\|a \times b\|}_{\text{area of base}} \cdot \text{height}$

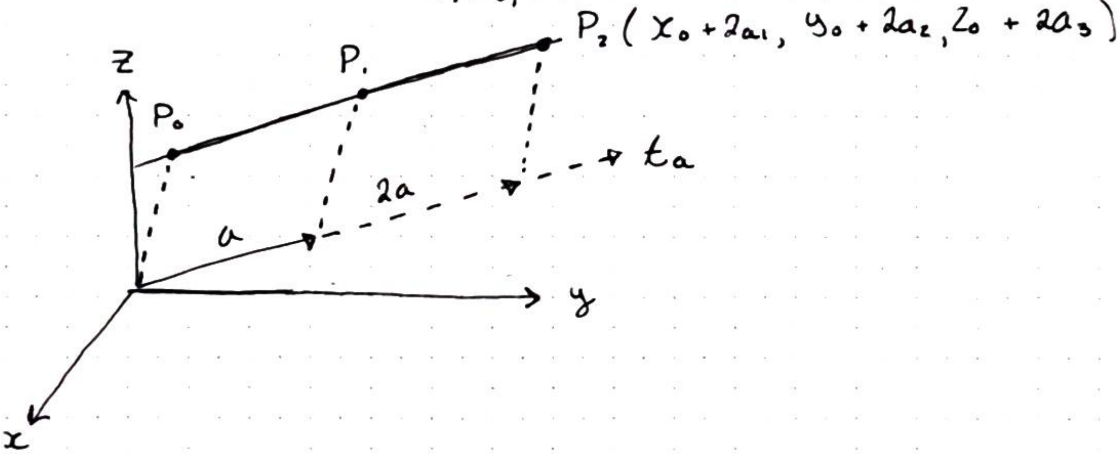
$= \|a \times b\| \cdot \text{comp}_{a \times b}\, c = \frac{c \cdot (a \times b)}{\cancel{\|a \times b\|}} \cancel{\|a \times b\|}$

$\Rightarrow$ Volume $= c \cdot (a \times b)$

Equations of Lines and Planes



Lines: Set up: we want to write the equation of a line that contains a given point $P_0(x_0, y_0, z_0)$ and which has the dir. of a vector $a = \langle a_1, a_2, a_3 \rangle$



Answer: Equation of this line:

$$\left.\begin{aligned} x &= x_0 + ta_1 \\ y &= y_0 + ta_2 \\ z &= z_0 + ta_3 \end{aligned}\right\} \text{Parametric eq'n of line}$$

(t = parameter)

If we eliminate t:

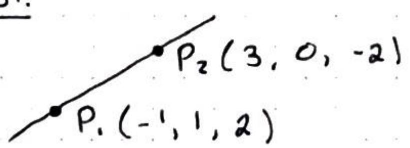
$$\underbrace{t = \frac{x - x_0}{a_1} \; ; \; t = \frac{y - y_0}{a_2} \; ; \; t = \frac{z - z_0}{a_3}}$$

$$\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}$$

(symmetric equation of line)

Ex. Find the equation of the line that passes through $P_1(-1, 1, 2)$ and $P_2(3, 0, -2)$

Sol.



Line < Point $P_1(\overset{x_0}{-1}, \overset{y_0}{1}, \overset{z_0}{2})$
direction vector

$a = \overrightarrow{P_1P_2}$

$= \langle 3-(-1), 0-1, -2-2 \rangle$

$= \langle \underset{a_1}{4}, \underset{a_2}{-1}, \underset{a_3}{-4} \rangle$

Equation of line:

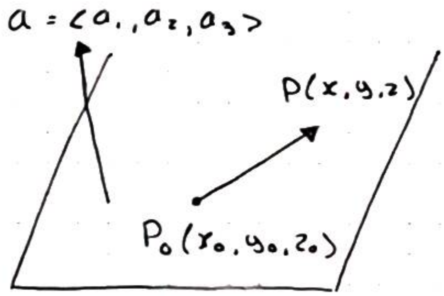
$$x = 1 + t \cdot 4 = 1 + 4t$$
$$y = 1 + t(-1) = 1 - t$$
$$z = 1 + t(-4) = 1 - 4t$$

<u>Planes</u>: <u>Set up</u>: We will write the equation of a plane that passes through a given point $P_0(x_0, y_0, z_0)$ and has

$$a = \langle a_1, a_2, a_3 \rangle = a_1 i + a_2 j + a_3 k$$

as a normal vector.

<u>Answer</u>:



Vector $a = \langle a_1, a_2, a_3 \rangle$ is perp. to $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$

$$\therefore a_1(x - x_0) + a_2(y - y_0) + a_3(z - z_0) = 0$$

dot product

$$a \cdot \overrightarrow{P_0P} = 0$$

$$a_1 x + a_2 y + a_3 z - \underbrace{(a_1 x_0 + a_2 y_0 + a_3 z_0)}_{\text{number}} = 0$$

<u>Ex</u>. Find the equation of the plane determined by $P(-1, -2, 0)$, $Q(1, 0, -1)$ and $R(2, 1, 0)$

<u>Sol</u>: Plane: Point: $P(\underset{x_0}{-1}, \underset{y_0}{-2}, \underset{z_0}{0})$

normal vector: $n = \overbrace{\overrightarrow{PQ} \times \overrightarrow{PR}}^{= 3i - 3j + 0k}$

$$\overrightarrow{PQ} = \langle 1 - (-1), 0 - (-2), -1 - 0 \rangle = \langle 2, 2, -1 \rangle$$
$$\overrightarrow{PR} = \langle 2 - (-1), 1 - (-2), 0 - 0 \rangle = \langle 3, 3, 0 \rangle$$

$$n = \begin{vmatrix} i & j & k \\ 2 & 2 & -1 \\ 3 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} i - \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} j + \begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} k$$

$$= 3i - 3j + 0k$$

Eq. of plane: $\underbrace{3}_{a_1}(x - \underbrace{(-1)}_{x_0}) + \underbrace{(-3)}_{a_2}(y - \underbrace{(-2)}_{y_0}) + \underbrace{(0)}_{a_3}(z - \underbrace{0}_{z_0})$ $= 0$

$\therefore\ 3x - 3y + 3 - 6 = 0$

$3x - 3y - 3 = 0$

$\boxed{x - y - 1 = 0}$

<u>Remark:</u> We see that planes have equations of the form

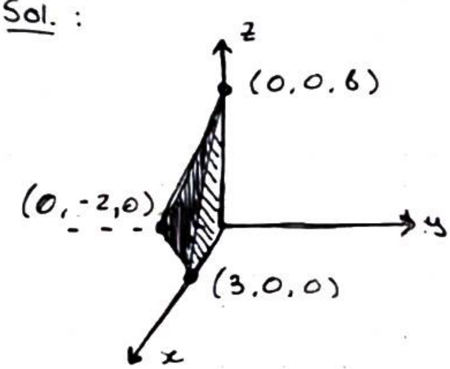
$$a_1 x + a_2 y + a_3 z - \underbrace{d}_{\text{number}} = 0$$

<u>Ex.</u>: $2x - 3y + 1z - 6 = 0$

$a = \langle 2, -3, 1 \rangle$ = normal vector to this plane

Draw the plane

<u>Sol.</u>:



We will find the intercepts of $2x - 3y + z - 6 = 0$ with the axes:

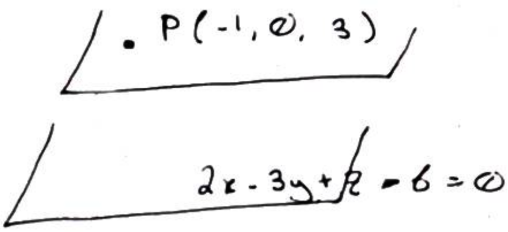
with x-axes: $y = 0$, $z = 0$, $2x - 3y + z - 6 = 0$ $\Big\}$ $2x - 6 = 0$, $x = 3$

with y-axes: $x = 0$, $z = 0$, $2x - 3y + z - 6 = 0$ $\Big\}$ $y = -2$

with z-axes: $z = 6$

<u>Ex.</u> What is the equation of the plane that passes through the point $P(-1, 0, 3)$ and is parallel to the plane $2x - 3y + z - 6 = 0$

<u>Sol.</u>



Plane $\Big\langle$ Point $P(-1, 0, 3)$ (x_0, y_0, z_0); normal vector $a = \langle 2, -3, 1 \rangle$ (a_1, a_2, a_3)

Eq. plane: $\underset{a_1}{2}(x - \underset{x_0}{(-1)}) \underset{a_2}{-3}(y - \underset{y_0}{0}) + \underset{a_3}{1}(z - \underset{z_0}{3}) = 0$

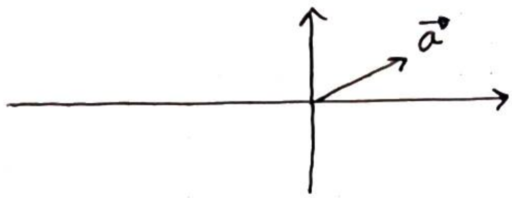
$2x - 3y + z + 2 - 3 = 0$

$2x - 3y + z - 1 = 0$

7.6 - Vector spaces

JAN. 19/18
APPLIED ANAL.

2 - space $\mathbb{R}^2$
3 - space $\mathbb{R}^3$
n - space $\mathbb{R}^n$



a vector in n-space is any ordered n-tuple $\vec{a} = \langle a_1, a_2 .. a_n \rangle$

IF $\vec{a} = \langle a_1, a_2, \dots, a_n \rangle$
$\vec{b} = \langle b_1, b_2, \dots, b_n \rangle$

$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, \dots, a_n + b_n \rangle$

$k\vec{a} = \langle ka_1, ka_2, \dots, ka_n \rangle$

The length of $\vec{a}$

$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$

The dot product of $\vec{a}$ and $\vec{b}$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

$\vec{a}$ and $\vec{b}$ are orthogonal $\iff \vec{a} \cdot \vec{b} = 0$

$\mathbb{R}^4$: (a_1, a_2, a_3, a_4)
priced stock 1

$\mathbb{R}^4$: $(\underbrace{a_1, a_2, a_3}_{\text{location of an object}}, a_4)$ → time

Vector space : a set of elements on which two operations are defined, one called vector addition and the other called scalar multiplication, and the following 10 properties are satisfied:

(i) IF $\vec{x}, \vec{y} \in V$, $\vec{x} + \vec{y} \in V$

(ii) For all $\vec{x}, \vec{y} \in V$, $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
(commutative law)

(iii) For all $\vec{x}, \vec{y}, \vec{z} \in V$
$\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + z$

(iv) There is a unique vector $\vec{0} \in V$
$\vec{0} + \vec{x} = \vec{x} = \vec{x} + \vec{0}$ for all $\vec{x} \in V$

(v) For each $\vec{x} \in V$ there exists a vector
$-\vec{x} + V$ s.t.
$-\vec{x} + (-\vec{x}) = (-\vec{x}) + \vec{x} = 0$

(vi) IF $\vec{x} \in V$, k is a scalar, $k\vec{x} \in V$

(vii) $k(\vec{x} + \vec{y}) = k\vec{x} + k\vec{y}$

(viii) $(k_1 + k_2)\vec{x} = k\vec{x} + k_2\vec{x}$

(ix) $k(k_2\vec{x}) = (k_1 k_2)\vec{x}$

(x) $1\vec{x} = \vec{x}$ for all $\vec{x} \in V$

Example $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$ are vector spaces under ordinary addition and multiplication by real numbers

Ex Determine whether the sets

(a) $V = \{1\}$ and (b) $V = \{0\}$

under ordinary addition and multiplication by real numbers

Solution (a) $V = \{1\}$

(i) If $\vec{x}, \vec{y} \in V$, $\vec{x} = 1$, $\vec{y} = 1$

$\vec{x} + \vec{y} = 1 + 1 = 2$ (and is <u>not</u> in V)

(i) Fails, so V is not in vector space

(b) $V = \{0\}$

(i) If $\vec{x}, \vec{y} \in V$, $\vec{x} = 0$, $\vec{y} = 0$

$\vec{x} + \vec{y} = 0$

(ii) For $\vec{x}, \vec{y} \in V$, $\vec{x} = 0$, $\vec{y} = 0$

$\vec{x} + \vec{y} = 0 + 0 = \vec{y} + \vec{x}$

⋮

(x) All of the 10 properties are satisfied (omit)

it is a vector space.

Example V - the set of all positive numbers

Define $\vec{x}, \vec{y} \in V$, $\vec{x} = x > 0$, $\vec{y} = y > 0$

$\vec{x} + \vec{y} = xy$ (ordinary multiplication)

For any scalar, $k\vec{x} = x^k$

Show that V is a vector space under the operation above.

Solution (i) For $\vec{x} = x$, $\vec{y} = y$ in V

$\vec{x} + \vec{y} = xy > 0 \quad \therefore \vec{x} + \vec{y} \in V$

(ii) For $\vec{x} = x$, $\vec{y} = y$ in V

$\vec{x} + \vec{y} = xy = yx = \vec{y} + \vec{x}$

(iii) For $\vec{x} = x$, $\vec{y} = y$, $\vec{z} = z$ in V

$\vec{x} + (\vec{y} + \vec{z}) = x(yz) = (xy)z = (\vec{x} + \vec{y}) + \vec{z}$

(iv) Let $\vec{0} = 1 \in V$. Then, for $\vec{x}$ in V

$\vec{0} + \vec{x} = 1 \cdot x = x = \vec{x} = \vec{x} + \vec{0}$

(v) For each $\vec{x} = x$ in V, let $-\vec{x} = 1/x$

$\vec{x} + (-\vec{x}) = x \cdot (1/x) = 1 = \vec{0}$

$(-\vec{x}) + \vec{x} = (1/x) \cdot x = 1 = \vec{0}$

(vi) If $\vec{x} = x$ in V, k is a scalar

$k\vec{x} = x^k > 0$ is in V

(vii) $k(\vec{x} + \vec{y}) = (xy)^k = x^k y^k = k\vec{x} + k\vec{y}$ ... etc. (to (x))

Example : P_3 - the set of all polynomials of degree 3 or less
P_3 is a vector space under ordinary addition of Polys and scalar multiplication.

$P3: \; a_3x^3 + a_2x^2 + a_1x + a$
(verify the 10 properties are satisfied)

...at can't be broken

...stances

... is neither ...ated or destroyed

F... ust: a given compound always contains the same proportion of elements by mass

- Dalton : when two elements form a series of compounds

Modern atomic theory

Nucleus - positively charged, dense centre
 Protons - positively charged, magnitude as an electron
 Neutrons - neutral particles, mass similar to proton

- # electrons = # protons.
- on periodic table, top # represents protons
- Alkali metals (except H - group 1A)
  - chemically reactive
- Alkaline Earth Metals (group 2A)
- Halogens (Group 7A)
- Noble gases (Group 8A)
  - generally non-reactive
- Noble metals - generally unreactive compared to other metals
- isotopes : atoms with the same number of protons, but different number of neutrons