JAN. 12/18

APPLIED ANAL

7.3 Dot Product

Definition: (1) $\vec{a} = \langle a_1, a_2 \rangle$, $\vec{b} = \langle b_1, b_2 \rangle$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

(2) $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Ex. IF $\vec{a} = 2i - 3j + 5k$, $\vec{b} = -\frac{1}{2} i + 2j - 3k$

$\vec{a} \cdot \vec{b} = (2)(-\frac{1}{2}) + (-3)(2) + (5)(-3) = -22$

Thm 7.3.1 (Properties)

(i) $\vec{a} \cdot \vec{b} = 0$ if $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

(ii) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(iii) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

(iv) $\vec{a} \cdot (k\vec{b}) = (k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})$

(v) $\vec{a} \cdot \vec{a} \geq 0$

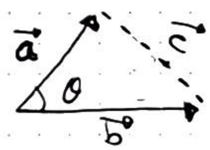
(vi) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

$$\left(\begin{aligned} \vec{a} \cdot \vec{a} &= a_1 a_1 + a_2 a_2 + a_3 a_3 \\ &= a_1^2 + a_2^2 + a_3^2 \end{aligned} \right)$$

Thm. 7.3.2 (Alternative Form)

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos \theta$

Where θ is the angle between $\vec{a}$ and $\vec{b}$



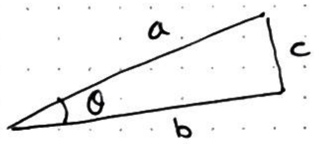
Proof Let $\vec{a} = a_1 i + a_2 j + a_3 k$

$\vec{b} = b_1 i + b_2 j + b_3 k$

Then $\vec{c} = \vec{b} - \vec{a} = (b_1 - a_1) i + (b_2 - a_2) j + (b_3 - a_3) k$

Consider:

Law of Cosines



$(c^2 = a^2 + b^2 - 2ab \cos \theta)$

By the law of cosines,

$\|\vec{c}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\| \|\vec{b}\| \cos \theta$

$2\|\vec{a}\| \|\vec{b}\| \cos \theta = \|\vec{a}\|^2 + \|\vec{b}\|^2 - \|\vec{c}\|^2$

$= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2)$

$= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - \cdots$

$\cdots [(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2]$

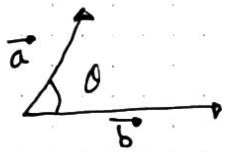
$= (\cancel{a_1^2} + \cancel{a_2^2} + \cancel{a_3^2}) + (\cancel{b_1^2} + \cancel{b_2^2} + \cancel{b_3^2}) - \cdots$

$\cdots [(\cancel{a_1^2} + 2a_1b_1 + \cancel{b_1^2}) + (\cancel{a_2^2} - 2a_2b_2 + \cancel{b_2^2}) + (\cancel{a_3^2} - 2a_3b_3 + \cancel{b_3^2})$

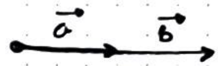
$\Rightarrow 2a_1b_1 + 2a_2b_2 + 2a_3b_3$

$\|\vec{a}\| \, \|\vec{b}\| \cos\theta = \vec{a} \cdot \vec{b}$

Angle between $\vec{a}$ and $\vec{b}$



(1) $\theta = 0$



$\cos\theta = 1$

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot (1)$

(2) $0 < \theta < \pi/2$, $\cos\theta > 0$,

$\vec{a} \cdot \vec{b} > 0$

(3) $\theta = \pi/2$, $\cos(\pi/2) = 0$, $\vec{a} \cdot \vec{b} = 0$

(4) $\pi/2 < \theta < \pi$,

, $\cos\theta < 0$

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta < 0$

Thm. 2.3.3 Two non-zero vectors $\vec{a}$ and $\vec{b}$ are orthogonal $\Leftrightarrow \vec{a} \cdot \vec{b} = 0$

Ex: If $\vec{a} = 2i - 3j + k$, $\vec{b} = 2i + 4j + 8k$

$\vec{a} \cdot \vec{b} = (2)(2)i + (-3)(4)j + (1)(8)k$

$= 4 - 12 + 8 = 0$

$\vec{a}$ and $\vec{b}$ are orthogonal (90°)

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta$

$\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|} \rightsquigarrow \theta = \cos^{-1}\left(\dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|}\right)$

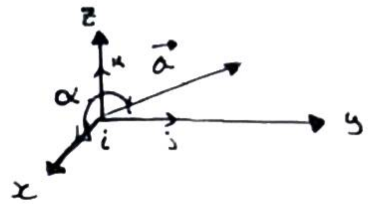
Ex. Find the angle between

$\vec{a} = 2i + 3j + k$ and $\vec{b} = -i + 5j + k$

$\vec{a} \cdot \vec{b} = (2)(-1) + (3)(5) + (1)(1) = 14$

$\left.\begin{aligned} \|\vec{a}\| &= \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14} \\ \|\vec{b}\| &= \sqrt{(-1)^2 + 5^2 + 1^2} = \sqrt{27} \end{aligned}\right\} \theta = \cos^{-1}\left(\dfrac{14}{\sqrt{14} \cdot \sqrt{27}}\right) \approx 0.77$ radian

Direction Cosines



Let $\vec{a} = a_1 i + a_2 j + a_3 k$

α = the angle between $\vec{a}$ and i

β = ... $\vec{a}$ and j

γ = ... $\vec{a}$ and k

then; $\cos\alpha = \dfrac{\vec{a_1} \cdot i}{\|\vec{a_1}\| \cdot \|\vec{i}\|} = \dfrac{\vec{a_1}}{\|\vec{a_1}\|}$

$\cos\beta = \dfrac{\vec{a_2} \cdot j}{\|\vec{a_2}\| \cdot \|j\|} = \dfrac{\vec{a_2}}{\|\vec{a_2}\|}$

$\cos\gamma = \dfrac{\vec{a_3} \cdot k}{\|\vec{a_3}\| \cdot \|k\|} = \dfrac{\vec{a_3}}{\|\vec{a_3}\|}$

We say that $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are direction cosines of $\vec{a}$

$$\frac{\vec{a}}{\|\vec{a}\|} = \frac{a_1}{\|a_1\|} i + \frac{a_2}{\|a_2\|} j + \frac{a_3}{\|a_3\|} k$$

$$= \cos\alpha i + \cos\beta j + \cos\gamma k$$

Since the magnitude of $\dfrac{\vec{a}}{\|\vec{a}\|}$ is 1;

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Ex. Find the direction cosines for

$\vec{a} = 2i + 5j + 4k$

Solution $\cos\alpha = \dfrac{a_1}{\|a_1\|} = \dfrac{2}{\sqrt{2^2+5^2+4^2}} = 2/\sqrt{45}$

$\cos\beta = \dfrac{a_2}{\|a_2\|} \Rightarrow \dfrac{5}{\sqrt{45}}$ / $\cos\gamma = \dfrac{a_3}{\|a_3\|} \Rightarrow 4/\sqrt{45}$

Component of $\vec{a}$ on $\vec{b}$

$\vec{a} = a_1 i + a_2 j + a_3 k$

a_1 = component of $\vec{a}$ on i = $\vec{a} \cdot i$

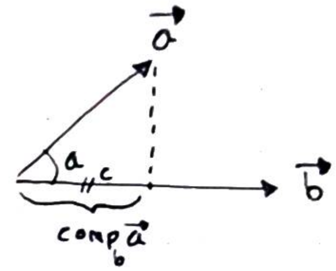
a_2 = component of $\vec{a}$ on j = $\vec{a} \cdot j$

a_3 = component of $\vec{a}$ on k = $\vec{a} \cdot k$

Let $\vec{b}$ be another vector

The component of $\vec{a}$ on $\vec{b}$

$$\text{Comp}_{\vec{b}}\,\vec{a} = \|\vec{a}\| \cos\theta$$



$$\text{Comp}_{\vec{b}}\,\vec{a} = \frac{\|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta}{\|\vec{b}\|}$$

$$\Rightarrow \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|}$$

<u>Ex.</u> Let $\vec{a} = 2i + 3j - 4k$

$\vec{b} = i + j + 2k$

Find $\text{Comp}_{\vec{b}}\,\vec{a}$

<u>Solution</u> : $\vec{a} \cdot \vec{b} = (2)(1) + (3)(1) + (-4)(2)$

$= -3$

$\|\vec{b}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{6}$

$\text{Comp}_{\vec{b}}\,\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|} = \dfrac{-3}{\sqrt{6}}$ ... cont'd.