7.3 Dot Product

Definition: (i) $\vec{a} = \langle a_1, a_2 \rangle$, $\vec{b} = \langle b_1, b_2 \rangle$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

(2) $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Ex. If $\vec{a} = 2i - 3j + 5k$, $\vec{b} = -\frac{1}{2} i + 2j - 3k$

$\vec{a} \cdot \vec{b} = (2)(-\frac{1}{2}) + (-3)(2) + (5)(-3) = -22$

Thm 7.3.1 (Properties)

(i) $\vec{a} \cdot \vec{b} = 0$ if $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

(ii) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(iii) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

(iv) $\vec{a} \cdot (k\vec{b}) = (k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})$

(v) $\vec{a} \cdot \vec{a} \geq 0$

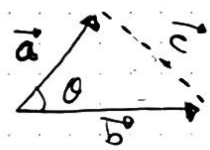
(vi) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

$$\left(\vec{a} \cdot \vec{a} = a_1 a_1 + a_2 a_2 + a_3 a_3 = a_1^2 + a_2^2 + a_3^2 \right)$$

Thm. 7.3.2 (Alternative Form)

$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta$

where θ is the angle between $\vec{a}$ and $\vec{b}$



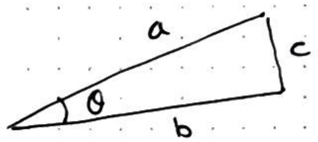
Proof Let $\vec{a} = a_1 i + a_2 j + a_3 k$

$\vec{b} = b_1 i + b_2 j + b_3 k$

Then $\vec{c} = \vec{b} - \vec{a} = (b_1 - a_1) i + (b_2 - a_2) j + (b_3 - a_3) k$

Consider:

Law of Cosines



$$\left(c^2 = a^2 + b^2 - 2ab\cos\theta \right)$$

By the law of cosines,

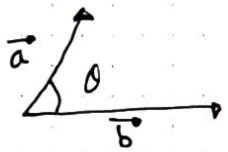
$$\|\vec{c}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$2\|\vec{a}\|\|\vec{b}\|\cos\theta = \|\vec{a}\|^2 + \|\vec{b}\|^2 - \|\vec{c}\|^2$$

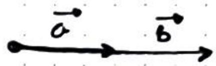
$$= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2)$$

$$= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - \cdots$$
$$\cdots \left[(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2\right]$$
$$= (\cancel{a_1^2} + \cancel{a_2^2} + \cancel{a_3^2}) + (\cancel{b_1^2} + \cancel{b_2^2} + \cancel{b_3^2}) - \cdots$$
$$\cdots \left[(\cancel{a_1^2} + 2a_1b_1 + \cancel{b_1^2}) + (\cancel{a_2^2} - 2a_2b_2 + \cancel{b_2^2}) + (\cancel{a_3^2} - 2a_3b_3 + \cancel{b_3^2})\right]$$
$$\Rightarrow 2a_1b_1 + 2a_2b_2 + 2a_3b_3$$
$$\|\vec{a}\|\,\|\vec{b}\| \cos\theta = \vec{a} \cdot \vec{b}$$

<u>Angle between $\vec{a}$ and $\vec{b}$</u>



(1) $\theta = 0$



$\cos\theta = 1$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot (1)$$

(2) $0 < \theta < \pi/2$, $\cos\theta > 0$,
$\vec{a} \cdot \vec{b} > 0$

(3) $\theta = \pi/2$, $\cos(\pi/2) = 0$, $\vec{a} \cdot \vec{b} = 0$

(4) $\pi/2 < \theta < \pi$,

, $\cos\theta < 0$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta < 0$$

<u>Thm. 2.3.3 Two non-zero vectors</u>
$\vec{a}$ and $\vec{b}$ are orthogonal $\iff \vec{a} \cdot \vec{b} = 0$

<u>Ex</u>: If $\vec{a} = 2i - 3j + k$, $\vec{b} = 2i + 4j + 8k$

$$\vec{a} \cdot \vec{b} = (2)(2)i + (-3)(4)j + (1)(8)k$$
$$= 4 - 12 + 8 = 0$$

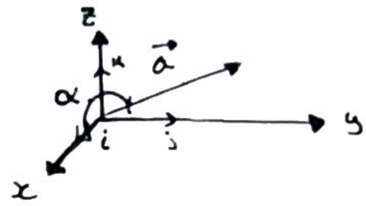
$\vec{a}$ and $\vec{b}$ are orthogonal (90°)

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta$$
$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|} \rightsquigarrow \theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|}\right)$$

<u>Ex.</u> Find the angle between
$\vec{a} = 2i + 3j + k$ and $\vec{b} = -i + 5j + k$

$$\vec{a} \cdot \vec{b} = (2)(-1) + (3)(5) + (1)(1) = 14$$
$$\left.\begin{aligned} \|\vec{a}\| &= \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14} \\ \|\vec{b}\| &= \sqrt{(-1)^2 + 5^2 + 1^2} = \sqrt{27} \end{aligned}\right\} \theta = \cos^{-1}\left(\frac{14}{\sqrt{14} \cdot \sqrt{27}}\right) \approx 0.77 \text{ radian}$$

Direction Cosines



Let $\vec{a} = a_1 i + a_2 j + a_3 k$

α = the angle between $\vec{a}$ and i

β = ... $\vec{a}$ and j

γ = ... $\vec{a}$ and k

then; $\cos\alpha = \dfrac{\vec{a}_1 \cdot i}{\|\vec{a}_1\| \cdot \|\vec{i}\|} = \dfrac{\vec{a}_1}{\|\vec{a}_1\|}$

$\cos\beta = \dfrac{\vec{a}_2 \cdot j}{\|\vec{a}_2\| \cdot \|j\|} = \dfrac{\vec{a}_2}{\|\vec{a}_2\|}$

$\cos\gamma = \dfrac{\vec{a}_3 \cdot k}{\|\vec{a}_3\| \cdot \|k\|} = \dfrac{\vec{a}_3}{\|\vec{a}_3\|}$

We say that $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are direction cosines of $\vec{a}$

$$\frac{\vec{a}}{\|\vec{a}\|} = \frac{a_1}{\|a_1\|} i + \frac{a_2}{\|a_2\|} j + \frac{a_3}{\|a_3\|} k$$

$$= \cos\alpha i + \cos\beta j + \cos\gamma k$$

Since the magnitude of $\dfrac{\vec{a}}{\|\vec{a}\|}$ is 1;

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Ex. Find the direction cosines for $\vec{a} = 2i + 5j + 4k$

Solution $\cos\alpha = \dfrac{a_1}{\|a_1\|} = \dfrac{2}{\sqrt{2^2+5^2+4^2}} = 2/\sqrt{45}$

$\cos\beta = \dfrac{a_2}{\|a_2\|} \Rightarrow \dfrac{5}{\sqrt{45}}$ / $\cos\gamma = \dfrac{a_3}{\|a_3\|} \Rightarrow 4/\sqrt{45}$

Component of $\vec{a}$ on $\vec{b}$

$\vec{a} = a_1 i + a_2 j + a_3 k$

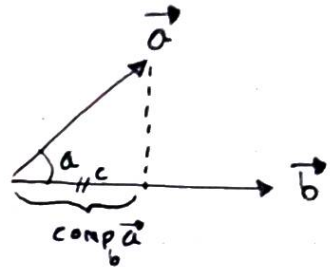
a_1 = component of $\vec{a}$ on i = $\vec{a} \cdot i$

a_2 = component of $\vec{a}$ on j = $\vec{a} \cdot j$

a_3 = component of $\vec{a}$ on k = $\vec{a} \cdot k$

Let $\vec{b}$ be another vector

The component of $\vec{a}$ on $\vec{b}$



$\text{Comp}_{\vec{b}} \vec{a} = \|\vec{a}\| \cos\theta$

$$\text{Comp}_{\vec{b}} \vec{a} = \frac{\|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta}{\|\vec{b}\|}$$

$$\Rightarrow \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|}$$

<u>Ex.</u> Let $\vec{a} = 2i + 3j - 4k$

$\vec{b} = i + j + 2k$

Find $\text{Comp}_{\vec{b}} \vec{a}$

<u>Solution</u> : $\vec{a} \cdot \vec{b} = (2)(1) + (3)(1) + (-4)(2)$

$= -3$

$\|\vec{b}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{6}$

$\text{Comp}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|} = \frac{-3}{\sqrt{6}}$ ... cont'd.

Cross Product

Another operation with vectors, but this time the resut is a Vector.

$$\left.\begin{aligned} a &= a_1 i + a_2 j + a_3 k \\ b &= b_1 i + b_2 j + b_3 k \end{aligned}\right\} \quad a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Side: 3×3 determinant

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \cdots$$

$$\cdots - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

then

$$a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$$= (a_2 b_3 - a_3 b_2) i - (a_1 b_3 - a_3 b_1) j + (a_1 b_2 - a_2 b_1$$

Proposition: Given two vectors a and b:

(1) $a \times b$ is perpendicular to both a and b

(2) $a \times b = - b \times a$

(3) $\|a \times b\| = \|a\| \cdot \|b\| \cdot \sin\theta$ (where θ is the angle between them)

Proof: (1) We want to check that $(a \times b) \cdot a = 0$

(and thus $a \times b$ is perpendicular to a)

$$(a \times b) \cdot a = (a_2 b_3 - a_3 b_2) a_1 - (a_1 b_3 - a_3 b_1) a_2 - \cdots$$
$$\cdots (a_1 b_2 - a_2 b_1) a_3$$

$$\Rightarrow \cancel{a_1 a_2 b_3} - \cancel{a_1 a_3 b_2} - \cancel{a_1 a_2 b_3} + \cancel{a_2 a_3 b_1} + \cancel{a_1 a_3 b_2} \cdots$$
$$\cdots - \cancel{a_2 a_3 b_1} = \boxed{0} \quad \blacksquare$$

(3) We can check that

$$\|a \times b\|^2 = \|a\|^2 \cdot \|b\|^2 \cdot \sin^2\theta$$

First: $\|a \times b\|^2 = (a_2 b_3 - a_3 b_2)^2 + (a_1 b_3 - a_3 b_1)^2 + (a_1 b_2 - a_2 b_1)^2$

$$\left[x = x_1 i + x_2 j + x_3 k \;:\; \|x\| = (x \cdot x)^{1/2} = \sqrt{x_1^2 + x_2^2 + x_3^2} \right]$$

Second: $\|a\|^2 \cdot \|b\|^2 (\sin^2\theta) = \|a\|^2 \cdot \|b\|^2 (1 - \cos^2\theta)$

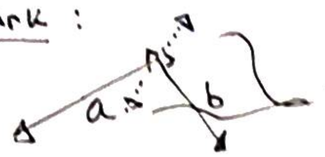
$$= \|a\|^2 \cdot \|b\|^2 - \|a\|^2 \cdot \|b\|^2 (1 - \cos^2\theta)$$

$$= \|a\|^2 \cdot \|b\|^2 - (a \cdot b)^2$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$

$$= \dots \blacksquare$$

Remark:



two potential cross-products

↳ Direction of $a \times b$ satisfies RHR.

Remark: (1) $i \times j$

$i = 1i + 0j + 0k$

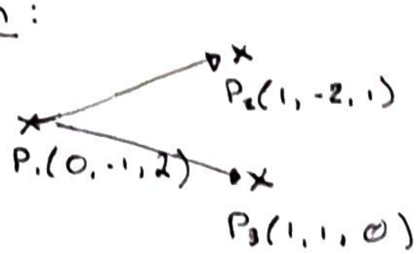
$j = 0i + 1j + 0k$

$$i \times j = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} i - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} j + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} k = 1k \checkmark$$

Example: Find a vector perpendicular to the plane determined by $P_1(0,-1,2)$, $P_2(1,-2,1)$ and $P_3(1,1,0)$

Solution:



$n = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$

$\overrightarrow{P_1P_2} = (1-0)i + [-2-(-1)]j + \dots$

$\dots (1-2)k = i - j - k$

$\overrightarrow{P_1P_3} = (1-0)i + [1-(-1)]j + \dots$

$\dots (0-2)k = i + 2j - 2k$

$$n = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ 1 & 2 & -2 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} -1 & -1 \\ 2 & -2 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} j + \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} k \Rightarrow 4i + j + 3k$$

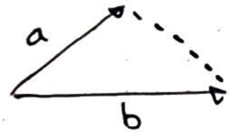
Other geometric applications of cross products

(1) Remember: $\|a \times b\| = \underbrace{\|a\| \cdot \|b\| \cdot \sin\theta}$ area of parallelogram formed by a and b

height $= \|a\| \cdot \sin\theta$

length of base $= \|b\|$

(2)

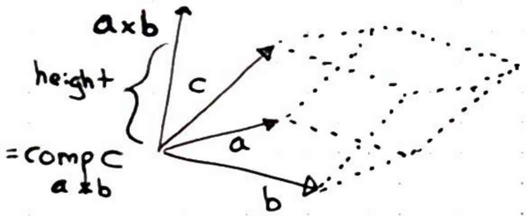


Area of the triangle formed by vectors a and b

$= \frac{1}{2}\|a \times b\|$

(3) Volume of the Parallelipiped determined by 3 vectors

$\underline{\underline{a}}$, $\underline{\underline{b}}$ and $\underline{\underline{c}}$



$V = (\text{area of base}) \times \text{height}$

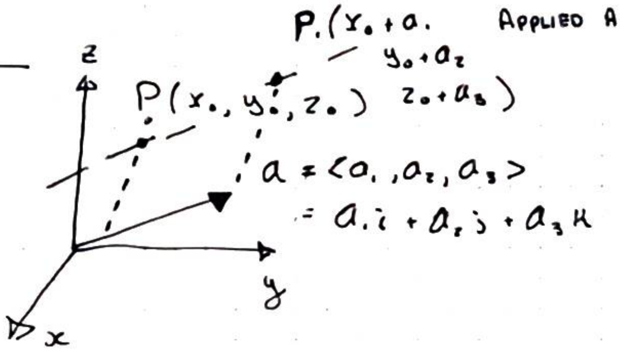
$= \underbrace{\|a \times b\|}_{\text{area of base}} \cdot \text{height}$

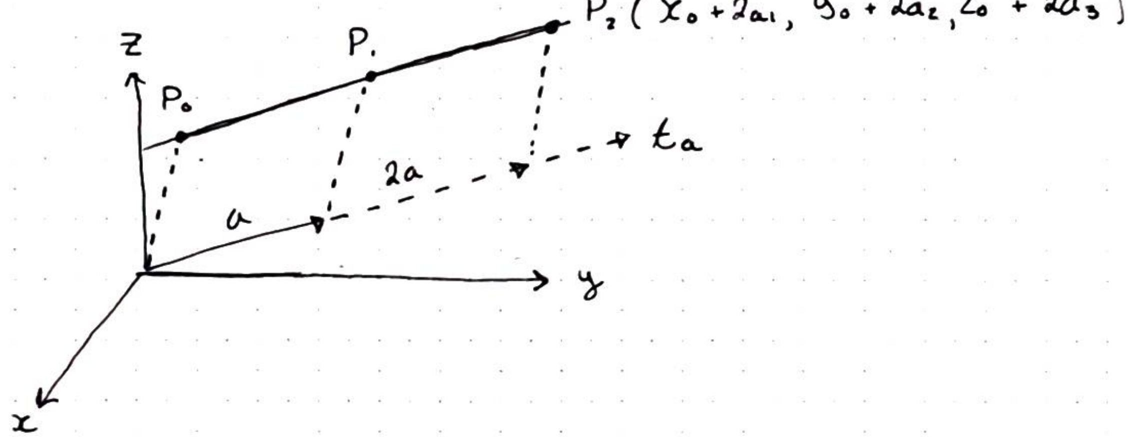
$= \|a \times b\| \cdot \text{comp}_{a\times b}\, c = \frac{c \cdot (a \times b)}{\cancel{\|a \times b\|}} \cancel{\|a \times b}$

$\Rightarrow$ Volume $= c \cdot (a \times b)$

Equations of Lines and Planes

Lines: Set up: we want to write the equation of a line that contains a given point $P_0(x_0, y_0, z_0)$ and which has the dir. of a vector $a = \langle a_1, a_2, a_3 \rangle$





Answer: Equation of this line:

$$\left.\begin{aligned} x &= x_0 + ta_1 \\ y &= y_0 + ta_2 \\ z &= z_0 + ta_3 \end{aligned}\right\} \text{Parametric eq'n of line}$$

(t = parameter)

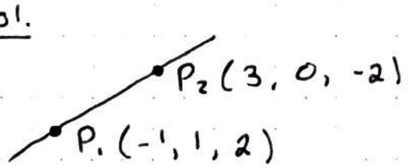
If we eliminate t: $\underbrace{t = \dfrac{x - x_0}{a_1}\ ;\ t = \dfrac{y - y_0}{a_2}\ ;\ t = \dfrac{z - z_0}{a_3}}$

$$\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}$$

(Symmetric equation of line)

Ex. Find the equation of the line that passes through $P_1(-1, 1, 2)$ and $P_2(3, 0, -2)$

Sol.



Line $\Big<$ Point $P_1(\overset{x_0}{-1}, \overset{y_0}{1}, \overset{z_0}{2})$; direction vector $a = \overrightarrow{P_1P_2}$

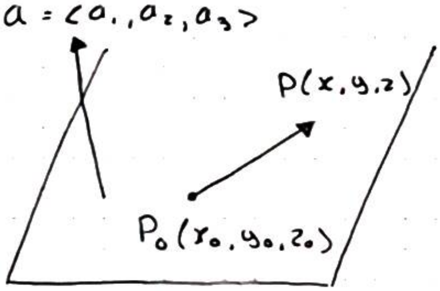
$= \langle 3-(-1), 0-1, -2-2 \rangle$

$= \langle 4, -1, -4 \rangle$
$\quad a_1 \quad a_2 \quad a_3$

Equation of line:
$x = 1 + t \cdot 4 = 1 + 4t$
$y = 1 + t(-1) = 1 - t$
$z = 1 + t(-4) = 1 - 4t$

Planes: Set up: We will write the equation of a plane that passes through a given point $P_0(x_0, y_0, z_0)$ and has
$$a = \langle a_1, a_2, a_3 \rangle = a_1 i + a_2 j + a_3 k$$
as a normal vector.

Answer:



Vector $a = \langle a_1, a_2, a_3 \rangle$ is perp. to $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$

$\therefore a_1(x - x_0) + a_2(y - y_0) + a_3(z - z_0) = 0$

dot product

$a \cdot \overrightarrow{P_0P} = 0$

$$a_1 x + a_2 y + a_3 z - \underbrace{(a_1 x_0 + a_2 y_0 + a_3 z_0)}_{\text{number}} = 0$$

Ex. Find the equation of the plane determined by $P(-1, -2, 0)$, $Q(1, 0, -1)$ and $R(2, 1, 0)$

Sol: Plane:
- Point: $P(\underset{x_0}{-1}, \underset{y_0}{-2}, \underset{z_0}{0})$
- normal vector: $n = \overrightarrow{PQ} \times \overrightarrow{PR} = 3i - 3j + 0k$

$\overrightarrow{PQ} = \langle 1 - (-1), 0 - (-2), -1 - 0 \rangle = \langle 2, 2, -1 \rangle$

$\overrightarrow{PR} = \langle 2 - (-1), 1 - (-2), 0 - 0 \rangle = \langle 3, 3, 0 \rangle$

$$n = \begin{vmatrix} i & j & k \\ 2 & 2 & -1 \\ 3 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} i - \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} j + \begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} k$$

$$= 3i - 3j + 0k$$

Eq. of plane: $\underbrace{3}_{a_1}(x - \underbrace{(-1)}_{x_0}) + \underbrace{(-3)}_{a_2}(y - \underbrace{(-2)}_{y_0}) + \underbrace{(0)}_{a_3}(z - \underbrace{0}_{z_0 = 0})$

$\therefore\ 3x - 3y + 3 - 6 = 0$

$3x - 3y - 3 = 0$

$\boxed{x - y - 1 = 0}$

<u>Remark:</u> We see that planes have equations, of the form

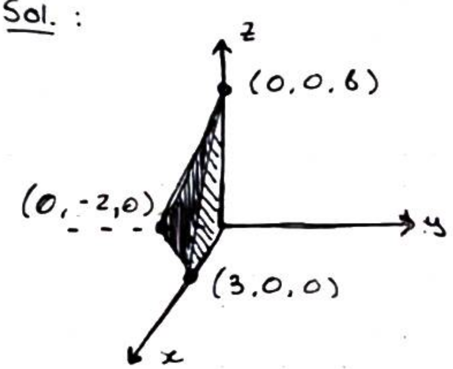
$$a_1 x + a_2 y + a_3 z - \underbrace{d}_{\text{number}} = 0$$

<u>Ex.</u>: $2x - 3y + 1z - 6 = 0$

$a = \langle 2, -3, 1 \rangle$ = normal vector to this plane

Draw the plane

<u>Sol.</u>:



We will find the intercepts of $2x - 3y + z - 6 = 0$ with the axes:

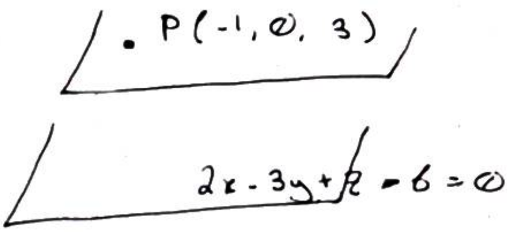
with x-axes: $\left.\begin{array}{l} y = 0 \\ z = 0 \\ 2x - 3y + z - 6 = 0 \end{array}\right\}$ $\begin{array}{l} 2x - 6 = 0 \\ x = 3 \end{array}$

with y-axes: $\left.\begin{array}{l} x = 0 \\ z = 0 \\ 2x-3y+z-6=0 \end{array}\right\}$ $y = -2$

with z-axes: $z = 6$

<u>Ex.</u> What is the equation of the plane that passes through the point $P(-1, 0, 3)$ and is parallel to the plane $2x - 3y + z - 6 = 0$

<u>Sol.</u>



Plane $\langle$ Point $P(\overset{x_0}{-1}, \overset{y_0}{0}, \overset{z_0}{3})$

normal vector

$a = \langle \underset{a_1}{2}, \underset{a_2}{-3}, \underset{a_3}{1} \rangle$

Eq. plane : $\underset{a_1}{2}(x - (\underset{x_0}{-1})) \underset{a_2}{-3}(y - \underset{y_0}{0}) + \underset{a_3}{1}(z - \underset{z_0}{3}) = 0$

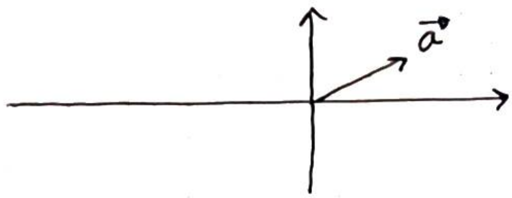
$2x - 3y + z + 2 - 3 = 0$

$2x - 3y + z - 1 = 0$

JAN. 19/18
APPLIED ANAL.

7.6 - Vector spaces

2 - space $\mathbb{R}^2$
3 - space $\mathbb{R}^3$
n - space $\mathbb{R}^n$



a vector in n-space is any ordered n-tuple $\vec{a} = \langle a_1, a_2, \ldots a_n \rangle$

IF $\vec{a} = \langle a_1, a_2, \ldots, a_n \rangle$
$\vec{b} = \langle b_1, b_2, \ldots, b_n \rangle$

$\vec{a} + \vec{b} = \langle a_1+b_1, a_2+b_2, \ldots, a_n+b_n \rangle$

$k\vec{a} = \langle ka_1, ka_2, \ldots, ka_n \rangle$

The length of $\vec{a}$

$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$

The <u>dot product</u> of $\vec{a}$ and $\vec{b}$

$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + \ldots + a_nb_n$

$\vec{a}$ and $\vec{b}$ are orthogonal $\iff \vec{a} \cdot \vec{b} = 0$

$\mathbb{R}^4: (\underset{\text{priced stock 1}}{a_1}, a_2, a_3, a_4)$

$\mathbb{R}^4: (\underbrace{a_1, a_2, a_3}_{\text{location of an object}}, \overset{\text{time}}{a_4})$

<u>Vector space</u>: a set of elements on which two operations are defined, one called vector addition and the other called scalar multiplication, and the following 10 properties are satisfied:

(i) IF $\vec{x}, \vec{y} \in V$, $\vec{x} + \vec{y} \in V$

(ii) For all $\vec{x}, \vec{y} \in V$, $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
(commutative law)

(iii) For all $\vec{x}, \vec{y}, \vec{z} \in V$
$\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + z$

(iv) There is a unique vector $\vec{0} \in V$
$\vec{0} + \vec{x} = \vec{x} = \vec{x} + \vec{0}$ for all $\vec{x} \in V$

(v) For each $\vec{x} \in V$ there exists a vector
$-\vec{x} + V$ s.t.
$-\vec{x} + (-\vec{x}) = (-\vec{x}) + \vec{x} = 0$

(vi) IF $\vec{x} \in V$, k is a scalar, $k\vec{x} \in V$

(vii) $k(\vec{x} + \vec{y}) = k\vec{x} + k\vec{y}$

(viii) $(k_1 + k_2)\vec{x} = k\vec{x} + k_2\vec{x}$

(ix) $k(k_2\vec{x}) = (k_1k_2)\vec{x}$

(x) $1\vec{x} = \vec{x}$ for all $\vec{x} \in V$

Example $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$ are vector spaces under ordinary addition and multiplication by real numbers

Ex Determine whether the sets

(a) $V = \{1\}$ and (b) $V = \{0\}$

under ordinary addition and multiplication by real numbers

Solution (a) $V = \{1\}$

(i) If $\vec{x}, \vec{y} \in V$, $\vec{x} = 1$, $\vec{y} = 1$

$\vec{x} + \vec{y} = 1 + 1 = 2$ (and is not in V)

(i) Fails, so V is not in vector space

(b) $V = \{0\}$

(i) If $\vec{x}, \vec{y} \in V$, $\vec{x} = 0$, $\vec{y} = 0$

$\vec{x} + \vec{y} = 0$

(ii) For $\vec{x}, \vec{y} \in V$, $\vec{x} = 0$, $\vec{y} = 0$

$\vec{x} + \vec{y} = 0 + 0 = \vec{y} + \vec{x}$

⋮

(x) All of the 10 properties are satisfied (omit)

it is a vector space.

Example V - the set of all positive numbers

Define $\vec{x}, \vec{y} \in V$, $\vec{x} = x > 0$, $\vec{y} = y > 0$

$\vec{x} + \vec{y} = xy$ (ordinary multiplication)

For any scalar, $k\vec{x} = x^k$

Show that V is a vector space under the operation above.

Solution (i) For $\vec{x} = x$, $\vec{y} = y$ in V

$\vec{x} + \vec{y} = xy > 0 \quad \therefore \vec{x} + \vec{y} \in V$

(ii) For $\vec{x} = x$, $\vec{y} = y$ in V

$\vec{x} + \vec{y} = xy = yx = \vec{y} + \vec{x}$

(iii) For $\vec{x} = x$, $\vec{y} = y$, $\vec{z} = z$ in V

$\vec{x} + (\vec{y} + \vec{z}) = x(yz) = (xy)z = (\vec{x} + \vec{y}) + \vec{z}$

(iv) Let $\vec{0} = 1 \in V$. Then, for $\vec{x}$ in V

$\vec{0} + \vec{x} = 1 \cdot x = x = \vec{x} = \vec{x} + \vec{0}$

(v) For each $\vec{x} = x$ in V, let $-\vec{x} = 1/x$

$\vec{x} + (-\vec{x}) = x \cdot (1/x) = 1 = \vec{0}$

$(-\vec{x}) + \vec{x} = (1/x) \cdot x = 1 = \vec{0}$

(vi) If $\vec{x} = x$ in V, k is a scalar

$k\vec{x} = x^k > 0$ is in V

(vii) $k(\vec{x} + \vec{y}) = (xy)^k = x^k y^k = k\vec{x} + k\vec{y}$ ... etc. (to (x))

Example : P_3 - the set of all polynomials of degree 3 or less
P_3 is a vector space under ordinary addition of Polys and scalar multiplication.

P3 : $a_3x^3 + a_2x^2 + a_1x + a$
(verify the 10 properties are satisfied)

...at can't be broken

...ustances

... is neither ...eated or destroyed

F...ust: a given compound always contains the same proportion of elements by mass

- Dalton : when two elements form a series of compounds

Modern atomic theory

Nucleus - Positively charged, dense centre

Protons - Positively charged, magnitude as an electron

Neutrons - neutral particles, mass similar to proton

- # electrons = # protons.
- on periodic table, top # represents protons
- Alkali metals (except H - group 1A)
  - chemically reactive
- Alkaline Earth Metals (group 2A)
- Halogens (Group 7A)
- Noble gases (Group 8A)
  - generally non-reactive
- Noble metals - generally unreactive compared to other metals
- isotopes : atoms with the same number of Protons, but different number of neutrons

Examples of Vector spaces :

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APPLIED ANAL.

$\mathbb{R}$?

P - the set of polynomials

$(x^3 + 2x + 1) + (-x^3 + x^2) = x^2 + 2x + 1$

P_n - the set of polynomials of degree at most n
(including the zero polynomial)

Let V be a vector space. The subset W of V is a <u>subspace</u> if it is a vector space using the same addition and scalar multiplication.

- e.g. P_n is a subspace of P
- e.g. P_3 is a subspace of P_5
- e.g. $\{0\}$ and V are subspaces of V

THM: Let W be a subset of a vector space V.
Then W is a subspace of V if and only if

(i) $0 \in W$

(ii) if $w_1, w_2 \in W$ then $w_1 + w_2 \in W$ (closure under addition)

(iii) if $w \in E$, $\lambda \in \mathbb{R}$, then $\lambda \in \mathbb{R}$, then $\lambda w \in W$ (closure under scalar mult.)

- e.g. let $V = \mathbb{R}^3$, $W = \{ \langle a, b, 2a+3b \rangle : a, b \in \mathbb{R} \}$
is W a subspace of V?

(i) let $a = b = 0$. Then $\langle 0, 0, 0 \rangle \in W$

(ii) Take $\langle a_1, b_1, 2a_1 + 3b_1 \rangle$, $\langle a_2, b_2, 2a_2 + 3b_2 \rangle \in W$
$\langle a_1 + a_2, b_1 + b_2, 2(a_1 + a_2) + 3(b_1 + b_2) \rangle \in W$

(iii) Take $\langle a, b, 2a + 3b \rangle \in W$, $\lambda \in \mathbb{R}$
$\lambda \langle a, b, 2a + 3b \rangle = \langle \lambda a, \lambda b, 2\lambda a + 3\lambda b \rangle \in W$
It is a subspace

- e.g. let $V = \mathbb{R}^2$, $W = \{ \langle a, a^2 \rangle : a \in \mathbb{R} \}$

(i) $\langle 0, 0 \rangle \in W$

(ii) Take $\langle a_1, a_1^2 \rangle$, $\langle a_2, a_2^2 \rangle \in W$
$\langle a_1, a_2^2 \rangle + \langle a_2, a_2^2 \rangle = \langle a_1 + a_2, a_1^2 + a_2^2 \rangle$

As $a_1^2 + a_2^2 \neq (a_1 + a_2)^2$, this will not be in W in general

$\langle 1, 1 \rangle \in W$, $\langle 2, 4 \rangle \in W$ $\langle 1, 1 \rangle + \langle 2, 4 \rangle = \langle 3, 5 \rangle \in W$

W is <u>not</u> closed under addition, so not a subspace

Let V be a vector space, $v_1, \dots, v_k \in V$

A vector $v \in V$ is a <u>linear combination</u> of $v_1, \dots, v_k$ if $v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_k v_k$ for some $\lambda_1, \dots, \lambda_k \in \mathbb{R}$

- e.g. in $\mathbb{R}^2$, $3\langle 1,2\rangle - 4\langle 2,3\rangle = \langle -5,-6\rangle$

  $\langle -5,-6\rangle$ is a linear combination of $\langle 1,2\rangle$ and $\langle 2,3\rangle$

- e.g. Is $\langle 1,2,3\rangle$ a linear combination of $\langle -1,0,1\rangle$ and $\langle 2,0,6\rangle$?

  <u>No</u>: $\lambda_1\langle -1,0,1\rangle + \lambda_2\langle 2,0,6\rangle = \langle -\lambda_1 + 2\lambda_2, 0, \lambda_1 + 6\lambda_2\rangle \neq \langle 1,2,3\rangle$

Let V be a vector space, $v_1,\dots,v_k \in V$. These vectors are <u>linearly dependent</u> if there exist $\lambda_1,\dots,\lambda_k \in \mathbb{R}$, not all zero, so that $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_k v_k = 0$.
If not they are <u>linearly independent.</u>

- e.g. $\vec{i}, \vec{j}, \vec{k}$ in $\mathbb{R}^3$ are linearly independent

  Suppose $\lambda_1\vec{i} + \lambda_2\vec{j} + \lambda_3\vec{k} = \vec{0}$

  $\langle \lambda_1,\lambda_2,\lambda_3\rangle = \langle 0,0,0\rangle$. So $\lambda_1 = \lambda_2 = \lambda_3 = 0$

- e.g. in P, the vectors $1, x, x^2, x^3$ are linearly dependent

  Suppose $\lambda_1(1) + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3$ is the zero polynomial

  Then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$

- e.g. are $\langle 1,0,2\rangle, \langle 3,0,0\rangle, \langle 2,-1,8\rangle$ linearly independent in $\mathbb{R}^3$?

  Suppose $\lambda_1\langle 1,0,2\rangle + \lambda_2\langle 3,0,0\rangle + \lambda_3\langle 2,-1,8\rangle = \langle 0,0,0\rangle$

  $\cancel{\lambda_1} + 2\lambda_1 + 3\cancel{\lambda_2} + 2\cancel{\lambda_3} - \lambda_3 + 8\lambda_3 = \langle 0,0,0\rangle$

  $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$. They are linearly independent.

- e.g. in $\mathbb{R}^2$, $\langle 1,2\rangle, \langle 2,3\rangle, \langle -5,-6\rangle$ are linearly dep.

  $3\langle 1,2\rangle - 4\langle 2,3\rangle = \langle -5,-6\rangle$

  $3\langle 1,2\rangle - 4\langle 2,3\rangle - 1\langle -5,-6\rangle = \langle 0,0\rangle$

Let V be a vector space, $v_1,\dots,v_k = V$ such that $v_1,\dots,v_k$ <u>span</u> V if every vector in V is a linear combination of $v_1,\dots,v_k$.

- e.g. $\vec{i}, \vec{j}, \vec{k}$ span $\mathbb{R}^3$

  $\langle a_1, a_2, a_3\rangle = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

- e.g. $1, x, x^2, x^3, \dots, x^n$ span P_n

  $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n = a_0 1 + a_1 x + a_2 x^2 + \dots + a_n x^n$

- e.g. there is no finite set of vectors that spans P

If $v_1, \dots, v_k \in V$, then the span of $v_1, \dots, v_k$ is the set of all linear combinations of $v_1, \dots, v_k$. It is a subspace of V

(i) $0 = 0v_1 + 0v_2 + \dots + 0v_k$ is in the span

(ii) $(\lambda_1 v_1 + \dots + \lambda_k v_k) + (\mu_1 v_1 + \dots + \mu_k v_k)$

$= (\lambda_1 + \mu_1)v_1 + \dots + (\lambda_k + \mu_k)v_k$ is in the span.

(iii) $\mu(\lambda_1 v_1 + \dots + \lambda_k v_k) = \mu\lambda_1 v_1 + \dots + \mu\lambda_k v_k$ is in the span

Let V be a vector space, then $v_1, \dots, v_k \in V$ are said to be a basis for V if they are linearly independent and span V.

- e.g. $\vec{i}, \vec{j}, \vec{k}$ Form a basis for $\mathbb{R}^3$
- e.g. in $\mathbb{R}^n$, let $e_1 = \langle 1,0,0,0,\dots,0\rangle$, $e_2 = \langle 0,1,0,0,\dots,0\rangle$ $\dots$ $e_n = \langle 0,0,0,0,\dots,1\rangle$.

Then $e_1, \dots, e_n$ form the standard basis for $\mathbb{R}^n$.

- e.g. in P_n, $1, x, x^2, x^3, \dots, x^n$ is the standard basis
- e.g. P has no simple basis

If V is a vector space, then the number of vectors is a basis for V is fixed, and is called the dimension of V. $\dim V$. If V has no simple basis it is infinite-dimensional.

- e.g. $\dim \mathbb{R}^n = n$
- e.g. $\dim P_n = n+1$
- e.g. P is infinite dimensional
- e.g. $W = \{\langle a, b, 2a+3b\rangle = a, b \in \mathbb{R}\}$

$\langle a, b, 2a+3b\rangle = a\langle 1, 0, 2\rangle + b\langle 0, 1, 3\rangle$

$\langle 1, 0, 2\rangle$, $\langle 0, 1, 3\rangle$ span W

They are linearly independent and $\therefore \dim W = 2$

In any vector space V, v_1, v_2 and linearly independent provided neither is a scalar multiple of the other.

[ONLY use with vectors]

Suppose $\lambda_1 v_1 + \lambda_2 v_2 = 0$. If $\lambda_1 = 0$,

$\lambda_1 v_1 = -\lambda_2 v_2$, so $v_1 = \frac{-\lambda_2}{\lambda_1} v_2$

If $\lambda_2 \neq 0$, $\lambda_2 v_2 = -\lambda_1 v_1$

$v_2 = \frac{-\lambda_1}{\lambda_2} v_1$

Last time:

- Subspace, Subspace test
- linear combination
- linear dependence / independence
- span
- basis
- dimension

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Applied Anal.

Let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$. Then $\vec{v}_1, \ldots, \vec{v}_k$ are said to be an orthogonal set of vectors is $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$

- e.g. $\langle 1,1,0\rangle, \langle 1,-1,0\rangle, \langle 0,0,3\rangle$

hense that $\vec{v}_1, \ldots \vec{v}_k \in \mathbb{R}^n$ are an orthogonal set of vectors if they are an orthogonal set and each is a unit vector.

We can normalize an orthogonal set of nonzero vectors to obtain an orthogonal set.

- e.g. above example $(1/\sqrt{2})\langle 1,1,0\rangle, ((1/\sqrt{2})\langle 1,-1,0\rangle$ $(1/3)\langle 0,0,3\rangle$ is an orthogonal set.

Orthogonal means $\vec{v_i} \cdot \vec{v_j} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0 & i \neq j \end{cases}$

"Kronecker delta"

We know that if V is a subspace of $\mathbb{R}^n$ and $\vec{v}_1, \ldots, \vec{v}_k$ is a basis for V, then for any $\vec{v} \in V$ there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ so that $\vec{v} = \lambda_1 \vec{v}_1 + \ldots + \lambda_k \vec{v}_k$ but finding $\lambda_1, \ldots, \lambda_k$ can be a lot of work.

But if $\vec{v}_1, \ldots, \vec{v}_k$ is an orthanormal basis, then

$\vec{v} = (\vec{v} \cdot \vec{v}_1)\vec{v}_1 + (\vec{v} \cdot \vec{v}_2)\vec{v}_2 + \ldots + (\vec{v} \cdot \vec{v}_k)\vec{v}_k$

Suppose $\vec{v} = \lambda_1 \vec{v}_1 + \ldots + \lambda_k \vec{v}_k$

Then $\vec{v} \cdot \vec{v}_i = \lambda_1 \vec{v}_1 \cdot \vec{v}_i + \lambda_2 \vec{v}_2 \cdot \vec{v}_i + \ldots + \lambda_k \vec{v}_k \cdot \vec{v}_i = \lambda_i$

- e.g. $\vec{v}_1 = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0\rangle$ $\vec{v}_2 = \langle 1/\sqrt{2}, -1/\sqrt{2}, 0\rangle$

$\vec{v}_3 = \langle 0,0,1\rangle$

write $\langle 3,5,2\rangle$ as a linear combination

$\langle 3,5,2\rangle \cdot \langle 1/\sqrt{2}, 1/\sqrt{2}, 0\rangle = 8/\sqrt{2}$

$\langle 3,5,2\rangle \cdot \langle 1/\sqrt{2}, 1/\sqrt{2}, 0\rangle = -2/\sqrt{2}$

$\langle 3,5,2\rangle \cdot \langle 0,0,1\rangle = 2$

$\langle 3,5,2\rangle \cdot \frac{8}{\sqrt{2}}\vec{v}_1 - \frac{2}{\sqrt{2}}\vec{v}_2 + 2\vec{v}_3$

② ... Suppose we have a basis $\vec{u}_1, \dots, \vec{u}_k$ for a V or $\mathbb{R}$?

We use the Gram-Schmidt algorithm to obtain an orthonormal basis. We will obtain an orthanormal basis $\vec{v}_1, \dots, \vec{v}_k$ for V, then normailize we obtain an orthonormal basis $\vec{w}_1, \dots, \vec{w}_k$.

$$\vec{v}_1 = \vec{u}_1$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \cdot \vec{v}_1 - \frac{\vec{u}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \cdot \vec{v}_2$$

$$\vdots$$

$$\vec{v}_k = \vec{u}_k - \frac{\vec{u}_k \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \cdot \vec{v}_1 - \frac{\vec{u}_k \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 - \dots - \frac{\vec{u}_k \vec{v}_{k-1}}{\vec{v}_{k-1} \cdot \vec{v}_{k-1}} \vec{v}_{k-1}$$

At any stage, we can multiply $\vec{v}_i$ by a non-zero scalar.

- e.g. V is a subspace of $\mathbb{R}^3$ with basis $\vec{u}_1 = \langle 1, 3, 2 \rangle$ $u_2 = \langle 3, 2, 0 \rangle$. Find orthonormal basis for V.

$$\vec{v}_1 = \vec{u}_1 = \langle 1, 3, 2 \rangle$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \langle 3, 2, 0 \rangle - (9/14) \langle 1, 3, 2 \rangle$$

Replace $\vec{v}_2$ with $14\vec{v}_2$

$$\vec{v}_2 = 14\langle 3, 2, 0 \rangle - 9\langle 1, 3, 2 \rangle = \langle 33, 1, -18 \rangle$$

Normalize: $\vec{w}_1 = 1/\sqrt{14} \langle 1, 3, 2 \rangle$, $\vec{w}_2 = \frac{1}{\sqrt{1089 + 1 + 324}} \langle 33, 1, -18 \rangle$

- e.g. Let V be a subspace of $\mathbb{R}^4$ with the series $\vec{u}_1 = \langle 1, 1, 0, 0 \rangle$, $\vec{u}_2 = \langle 2, -1, 0, 0 \rangle$, $\vec{u}_3 = \langle 1, 1, -1, 0 \rangle$

Find an orthonormal basis

$$\vec{v}_1 = \vec{u}_1 = \langle 1, 1, 0, 0 \rangle$$

$$\vec{v}_1 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \langle 2, -1, 0, 0 \rangle - (1/2) \langle 1, 1, 0, 0 \rangle$$

Replace $\vec{v}_2$ with $2\vec{v}_2$

$$\vec{v}_2 = 2\langle 2, -1, 0, 0 \rangle - 1\langle 1, 1, 0, 0 \rangle = \langle 3, -3, 0, 0 \rangle$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \cdot \vec{v}_1 - \frac{\vec{u}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 = \langle 1, 1, -1, 0 \rangle - (2/2) \langle 1, 1, 0, 0 \rangle \dots$$

$$\dots (0/18) \langle 3, -3, 0, 0 \rangle$$

$$= \langle 0, 0, -1, 0 \rangle$$

Normalize $\vec{w}_1 = 1/\sqrt{2} \langle 1, 1, 0, 0 \rangle$ $\vec{w}_2 = (1/\sqrt{18}) \langle 3, -3, 0, 0 \rangle$

$$\vec{w}_3 = \langle 0, 0, -1, 0 \rangle$$

Chapter 8 - Matrices (8.1 - 8.10, 8.12, 8.14, 8.15)

Let m and n be positive integers. Then an $m \times n$ <u>matrix</u> is a rectangular array of numbers with m rows, n columns

- e.g. 2×3 matrix: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 12 & 0 \end{pmatrix}$ 3×2 matrix $\begin{pmatrix} -2 & 7 \\ 6 & 5 \\ 18 & 7 \end{pmatrix}$

We use capital letters to denote matrices.
If A is out matrix, then we write a_{ij} for the (ij)-entry that is, the entry in row i, column j

- e.g. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $a_{23} = 6$, $a_{31} = 7$

The <u>diagonal entries</u> are a_{ii}

- e.g. above, 1, 5, 9

A <u>square matrix</u> has the same number of rows as columns
A <u>column vector</u> is a matrix with 1 column
A <u>row vector</u> is a matrix with 1 row

- e.g. Column vector: $\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$, row vector $(-2, 1, 0, 4, 7)$

The $m \times n$ <u>zero matrix</u> has zeroes for all entries.
Denote the matrix O

We write $A = B$ if A and B are both $m \times n$ matrices and $a_{ij} = b_{ij}$ for all i, j

- e.g. $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \neq \begin{pmatrix} 5 & 3 \\ 1 & 2 \end{pmatrix}$

Let A and B be $m \times n$ matrices then their <u>sum</u> $A + B$ is the $m \times n$ matrix C so that $c_{ij} = a_{ij} + b_{ij}$ for all ij

- e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 7 & 2 & -1 \end{pmatrix} + \begin{pmatrix} -3 & 4 & 5 \\ 2 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1+(-3) & 2+4 & 3+5 \\ 7+2 & 0+2 & -1+4 \end{pmatrix} = \begin{pmatrix} -2 & 6 & 8 \\ 9 & 2 & 3 \end{pmatrix}$

If A and B do not have the same size their sum is undefined.

If A is an $m \times n$ matrix, its <u>negative</u> $-A$ is found by taking the negative of each entry.

- e.g. $-\begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 6 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & -1 & 2 \\ -3 & -2 & -6 \end{pmatrix}$

Properties of Matrix addition:

Let A, B, C be $m \times n$ matrices. Then

(i) $A+B$ is an $m \times n$ matrix

(ii) $(A+B)+C = A+(B+C)$ (associativity)

(iii) $A+B = B+A$ (commutivity)

(iv) $A + O = A$ (additive)

(v) $A + (-A) = O$ (additive merge)

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APPLIED ANAL.

→ Find all 3-Forms of the line through $(1,3,5)$ and $(2,-1,7)$

Vector: $(x, y, z) = (1,3,5) + t(1,-4,2)$

Parametric: $x = 1+t$, $y = 3-4t$, $z = 5+2t$

Symmetric: $\frac{x-1}{1} = \frac{y-3}{(-4)} = \frac{z-5}{2}$

→ A triangle has verticies $(1,2,0)$, $(2,1,-1)$, $(4,3,7)$
Find the area.

Side vectors: $\vec{a} = \langle 1,-1,-1 \rangle$
$\vec{b} = \langle 2,2,8 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -1 \\ 2 & 2 & 8 \end{vmatrix} \Rightarrow \begin{vmatrix} -1 & -1 \\ 2 & 8 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ 2 & 8 \end{vmatrix} j + \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} k$$

$$\Rightarrow \langle -6, -10, 4 \rangle$$

Area $= \frac{1}{2}\sqrt{(-6^2)+(-10)^2+(4)^2} = \frac{1}{2}\sqrt{150}$ ish

↳ cross products only exist in $\mathbb{R}^3$ space.
(the same question with

→ Let V be the set of all polynomials in the Form
$a + bx + (a+b)x^2$, $a, b \in \mathbb{R}$

Is V a subspace of P?

Let $a = b = 0$: $0 \leftarrow V$

$(a_1 + b_1x + (a_1+b_1)x^2) + (a_2 + b_2x + (a_2+b_2)x^2)$
$= (a_1+a_2) + (b_1+b_2)x + (a_1+b_1+a_2+b_2)x^2$
$= (a_1+a_2) + (b_1+b_2)x + ((a_1+a_2)+(b_1+b_2))x^2 \in V$

$\lambda(a + bx + (a+b)x^2)$
$= \lambda a + \lambda b x + (\lambda a + \lambda b)x^2 \in V$

V is a subspace

→ e.g. Let $V = \{\langle a, b, \|a+b\| \rangle, a,b \in \mathbb{R}\}$

is V a subspace of $\mathbb{R}^3$

Let $a = b = \emptyset = \langle 0,0,0 \rangle \in V$

$\langle a_1, b_1, |a_1+b_1| \rangle + \langle a_2, b_2 \rangle, |a_2+b_2| \rangle$

$= \langle a_1 + a_2, b_1 + b_2, |a_1+b_1| + |a_2+b_2| \rangle$

$\langle 1,1,2 \rangle + \langle -1,-1,2 \rangle = \langle 0,0,4 \rangle \notin V$

Not a subspace, not closed under addition

→ Are these polynomials linearly dependent or independent?

$3 + 4x^2, \ 2 - 7x + 2x^2, \ 6x^2$

$\lambda_1(3 + 4x^2) + \lambda_2(2 - 7x + 2x^2) + \lambda_3(6x^2) = \emptyset + \emptyset x + \emptyset x^2$

$(3\lambda_1 + 2\lambda_2) - (7\lambda_2)x + (4\lambda_1 + 2\lambda_2 + 6\lambda_3)x^2 = \emptyset + \emptyset x + \emptyset x^2$

$3\cancel{\lambda_1}^{0} + 2\cancel{\lambda_2}^{0} = \emptyset \quad / \quad -7\lambda_2 = \emptyset \quad / \quad 4\cancel{\lambda_1}^{0} + 2\cancel{\lambda_2}^{0} + 6\cancel{\lambda_3}^{0} = \emptyset$

$\lambda_1 = \emptyset \qquad \lambda_2 = \emptyset \qquad \lambda_3 = \emptyset$

$\therefore$ linearly <u>independent</u>

→ Let V be the subspace of $\mathbb{R}^3$ with basis

$\vec{u}_1 = \langle 1,1,1 \rangle, \ \vec{u}_2 = \langle 1,-3,1 \rangle, \ \vec{u}_3 = \langle 1,0,2 \rangle$

Find an orthogonal set for V

Graham-Schmidt : $\vec{V}_1 = \vec{U}_1 = \langle 1,1,1 \rangle$

$\vec{V}_2 = \vec{U}_2 - \dfrac{U_2 \cdot V_1}{V_1 \cdot V_1} = \langle 1,-3,1 \rangle - (-\tfrac{1}{3})\langle 1,1,1 \rangle$

Mult by 3: $\vec{V}_2 = 3\langle 1,-3,1 \rangle + \langle 1,1,1 \rangle = \langle 4,-8,4 \rangle$

Mult by $1/4$: $\vec{V}_2 = \langle 1,-2,1 \rangle$

$\vec{V}_3 = \vec{U}_3 - \dfrac{U_3 \cdot V_1}{V_1 \cdot V_1} V_1 - \dfrac{U_3 V_2}{V_2 V_2} V_2$

$\Rightarrow \langle 1,\emptyset,2 \rangle - (2/3)\langle 1,1,1 \rangle - (3/6)(1/2)\langle 1,-2,1 \rangle$

mult by 2 : $\vec{V}_3 = 2\langle 1,\emptyset,2 \rangle - 2\langle 1,1,1 \rangle - \langle 1,-2,1 \rangle$

$= \langle -1,0,1 \rangle$

Normalize : $(\tfrac{1}{\sqrt{3}})\langle 1,1,1 \rangle, (\tfrac{1}{\sqrt{6}})\langle 1,-2,1 \rangle, (\tfrac{1}{\sqrt{2}})\langle -1,0,1 \rangle$

- Last time: orthogonal / orthonormal set
  - orthonormal set
  - Graham - Schmitt
  - $m \times n$ matrix
  - Square matrix ($m = n$)
    → row vector, column vector
  - zero vector
  - Matrix addition

Let A be an $m \times n$ matrix, $\lambda \in \mathbb{R}$, then the scalar multiple λA is the $m \times n$ matrix B so that $b_{ij} = \lambda a_{ij}$ for all i, j

e.g. $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 0 & 6 \end{pmatrix}$, $\lambda A = \begin{pmatrix} 4 & 8 \\ 12 & -4 \\ 0 & 24 \end{pmatrix}$

Properties: let A and B be $m \times n$ matrices $\lambda, \mu \in \mathbb{R}$

(i) λA is an $m \times n$ matrix (closure under scalar mult.)

(ii) $\lambda(A+B) = \lambda A + \lambda B$ (distributive law)

(iii) $(\lambda + \mu)A = \lambda A + \mu A$ (" ")

(iv) $\lambda(\mu A) = (\lambda\mu)A$

(v) $1A = A$

The $m \times n$ matrices form a vector space

Let $A = (a_{11}, a_{12}, \dots, a_{1n})$ be a row vector $(1 \times n)$

Let $B = \begin{pmatrix} b_{11} \\ b_{21} \\ \dots \\ b_{n1} \end{pmatrix}$ be a column vector $(n \times 1)$

Then $AB = (a_{11}b_{11} + a_{12}b_{12} + \dots + a_{1n}b_{n1})$ (a 1×1 matrix)

- e.g. $(1, 3, 2)\begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} = ((1)(4) + (3)(5) + (2)(7)) = 33$

Let A be an $m \times n$ matrix and B be an $m \times n$ matrix

Then the <u>product</u> AB is the $m \times n$ matrix C so that

$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$

- e.g. $\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 4 & -3 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} (1)(2) + (3)(4) & (1)(1) + (3)(-3) & (1)(0) + (3)(2) \\ (2)(1) + (5)(4) & (2)(1) + (5)(-3) & (2)(0) + (5)(2) \end{pmatrix}$

  $\underbrace{2 \times 2 \quad 2 \times 3}$ → $\begin{pmatrix} 14 & -8 & 6 \\ 24 & -13 & 10 \end{pmatrix}$
  $\underline{2 \times 3}$

- e.g. $\begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 10 & 21 \end{pmatrix}$

  $\underbrace{2 \times 3 \quad 3 \times 2}$ $\quad \underline{2 \times 2}$

Properties : (i) $(AB)C = A(BC)$ — (associvity)

(ii) $A(B+C) = AB + AC$ — (distributive law)

(iii) $(B+C)A = BA + CA$ — (" ")

Matrix multiplication is <u>not</u> commutative!

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$AB = \emptyset \neq A = \emptyset$ or $B = \emptyset$

The $m \times n$ <u>identity matrix</u> I_n has all of the diagonal entries as "1", everything else is $\emptyset$

- e.g. $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

If A is an $m \times n$ matrix then $I_m A = A = AI_n$

Let A be an $m \times n$ matrix, then its' transpose A^T is the $m \times n$ matrix B so that $b_{ij} = a_{ji}$ for all i, j

- e.g. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

Properties : Let A, B be $m \times n$ matrices, C is $m \times$

$\lambda \in \mathbb{R}$

(i) $(A+B)^T = A^T + B^T$

(ii) $(\lambda A)^T = \lambda(A^T)$

(iii) $(A^T)^T = A$

(iv) $(AC)^T = C^T A^T$

Let A be a square matrix. Then A is the <u>upper triangular</u> matrix if all the entries below are zero.

- e.g. $\begin{pmatrix} 1 & 4 & -2 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

A is lower diagonal matrix is all entries above are zero.

- e.g $\begin{pmatrix} 3 & 0 \\ 1 & 4 \end{pmatrix}$

In a diagonal matrix, all entries off the diagonal are zero

- e.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

A **system of linear equations** has the form

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

This is a system of m equations in n unknowns

The x_i are the **variables**, the a_{ij} are the **coefficients** and the b_i are the **constants**

- e.g. $2x_1 + 5x_2 = 6$ 2 eqs, 2 unknowns
$-7x_1 + 2x_2 = 14$

A **solution** is a system is an n-tuple $(c_1, \dots, c_n)$ so that letting $x_1 = c_1$, $x_2 = c_2, \dots, x_n = c_n$, all equations are satisfied simultaneously.

To **solve** a system means to find all possible solutions

The system is **inconsistent** if there are no solutions

Otherwise, it is **consistent**, and there may be one solution or infinitely many.

The following **elementary row operations** can be performed without changing the solution to the system.

1) Multiply an equation by a non-zero constant
2) Add a multiple of one equation to another
3) Swap row equations

- e.g. solve
$$2x_1 + 6x_2 = 8 \quad (1)$$
$$4x_1 + 11x_2 = 1 \quad (2)$$

Multiply (1) by ($1/2$)
$$x_1 + 3x_2 = 4 \quad (1)$$
$$4x_1 + 11x_2 = 1 \quad (2)$$

Add -4(1) to (2)
$$x_1 + 3x_2 = 4 \quad (1)$$
$$-x_2 = -15 \quad (2)$$

Multiply (2) by -1
$$x_1 + 3x_2 = 4 \quad (1)$$
$$x_2 = 15 \quad (2)$$

Add -3(2) to (1)
$$x_1 = -41 \quad (1)$$
$$x_2 = 15 \quad (2)$$

$x_1 = -41$, $x_2 = 15$ (unique solution)

If our system of equations is:

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

3 ... then the <u>coefficient matrix</u> is

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

And the <u>augmented matrix</u> is

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} & b_m \end{array}\right)$$

The augmented matrix conveys all the information about the system. we can perform <u>elementary row operations</u> that do not change the solution to the system.

1) Multiply a row by a nonzero constant
2) Add a multiple of one row to another
3) swap two rows

- e.g. $2x_1 + 6x_2 = 8$ Form the augmented matrix:
$4x_1 + 11x_2 = 1$

$$\left(\begin{array}{cc|c} 2 & 6 & 8 \\ 4 & 11 & 1 \end{array}\right) \xrightarrow[\text{by } 1/2]{\text{Mult. } 1} \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 4 & 11 & 1 \end{array}\right) \xrightarrow[\text{to } 2]{\text{add } -4\, 1} \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & -1 & -15 \end{array}\right)$$

$$\xrightarrow[\text{by } -1]{\text{mult } 2} \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & 15 \end{array}\right) \xrightarrow[\text{to } 1]{\text{add } -3\, 2} \left(\begin{array}{cc|c} 1 & 0 & -41 \\ 0 & 1 & 15 \end{array}\right).$$

$\therefore \quad x_1 = -41$

$\quad\quad x_2 = 15$

- last time — invertible matrix.
  - inverse $AA^{-1}=A^{-1}A=I_n$.
  - $(A^{-1})^{-1}=A$, $(AB)^{-1}=B^{-1}A^{-1}$; $(A^T)^{-1}=(A^{-1})^T$.
  - $(A|I_n)\to(I_n|A^{-1})$

    $\searrow$ (zero row | ?), A not invertible.

  - adjoint matrix.
  - A invertible $\Leftrightarrow$ $\det(A)\neq 0$.

    In this case, $A^{-1}=\dfrac{1}{\det(A)}\operatorname{adj}(A)$

  - $AX=B$, A invertible $\Rightarrow$ $X=A^{-1}B$.
  - homogeneous system — trivial/nontrivial solution.
  - A square, then $AX=0$ has only the trivial solution

    $\Leftrightarrow$ A invertible.

If we have a system of equations $AX=B$ and A is invertible, we can use Cramer's rule.

For each i, let A_i be the matrix obtained by replacing column i of A with B.

Then $x_i = \dfrac{\det(A_i)}{\det(A)}$.

e.g. solve $\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_1 + 9x_2 &= 7 \end{aligned}$. $A = \begin{pmatrix} 1 & 2 \\ 3 & 9 \end{pmatrix}$, $\det(A) = 3$.

$A_1 = \begin{pmatrix} 5 & 2 \\ 7 & 9 \end{pmatrix}$, $\det(A_1) = 31$.

$A_2 = \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix}$, $\det(A_2) = -8$.

$x_1 = \frac{31}{3}$, $x_2 = \frac{-8}{3}$.

e.g. solve $\begin{aligned} x_1 \quad\;\; + x_3 &= 1 \\ x_1 + 2x_2 + x_3 &= 5 \\ x_1 + 2x_2 + 3x_3 &= 7 \end{aligned}$. $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$

$\det(A) = 1(-1)^{1+1}\det\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} + 1(-1)^{1+3}\det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

$= 4 + 0 = 4$.

$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 5 & 2 & 1 \\ 7 & 2 & 3 \end{pmatrix}$ $\det(A_1) = 1(-1)^{1+1}\det\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} + 1(-1)^{1+3}\det\begin{pmatrix} 5 & 2 \\ 7 & 2 \end{pmatrix}$

$= 4 + (-4) = 0$.

$A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 7 & 3 \end{pmatrix}$. $\det(A_2) = 1(-1)^{1+1}\det\begin{pmatrix} 5 & 1 \\ 7 & 3 \end{pmatrix} + 1(-1)^{1+2}\det\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} + 1(-1)^{1+3}\det\begin{pmatrix} 1 & 5 \\ 1 & 7 \end{pmatrix}$

$= 8 - (2) + 2 = 8$.

$A_3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 5 \\ 1 & 2 & 7 \end{pmatrix}$, $\det(A_3) = 1(-1)^{1+1}\det\begin{pmatrix} 2 & 5 \\ 2 & 7 \end{pmatrix} + 1(-1)^{1+3}\det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

$= 4 + 0 = 4$.

$x_1 = \frac{0}{4} = 0,\ x_2 = \frac{8}{4} = 2,\ x_3 = \frac{4}{4} = 1.$

Let A be an $n\times n$ matrix. A vector in $\mathbb{R}^n$ will be regarded as an $n\times 1$ column vector.

We say that a number λ is an eigenvalue for A if there is a nonzero vector $x \in \mathbb{R}^n$ s.t. $Ax = \lambda x$.

We say that x is an eigenvector corresponding to λ.

If we take all the eigenvectors corresponding to λ, including the zero vector, we obtain the eigenspace corresponding to λ.

Checking if x is an eigenvector of A is easy: calculate Ax and see if it is a scalar multiple of x

e.g. $A = \begin{pmatrix}1&2\\2&1\end{pmatrix}$. Are these eigenvectors? (i) $\begin{pmatrix}1\\1\end{pmatrix}$ (ii) $\begin{pmatrix}-1\\1\end{pmatrix}$ (iii) $\begin{pmatrix}3\\4\end{pmatrix}$.

(i) $\begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix} = 3\begin{pmatrix}1\\1\end{pmatrix}$, so it is an eigenvector corresponding to $\lambda = 3$.

(ii) $\begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} = -1\begin{pmatrix}-1\\1\end{pmatrix}$, " " " " " $\lambda = -1$.

(iii) $\begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}11\\10\end{pmatrix}$, $\begin{pmatrix}3\\4\end{pmatrix}$ is not an eigenvector.

If λ is an eigenvalue of A, then its eigenspace is a subspace of $\mathbb{R}^n$.

Indeed: $A(0) = 0 \overset{}{=} \lambda 0$, so 0 is in the eigenspace.

Let $Ax = \lambda x$, $Ay = \lambda y$. $A(x+y) = Ax + Ay = \lambda x + \lambda y$

$= \lambda(x+y)$.

$\therefore x+y$ is in the eigenspace.

Let $Ax = \lambda x$, $\mu \in \mathbb{R}$. Then $A(\mu x) = \mu Ax = \mu \lambda x = \lambda(\mu x)$.

$\therefore \mu x$ is in the eigenspace.

If we have an eigenvalue for A, we would like to describe its eigenspace. We can find a basis for the eigenspace.

e.g. $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Its eigenvalues are 0 and 2.

$\lambda = 2$: solve $Ax = 2x$, i.e. $Ax - 2x = 0$.

$Ax - 2Ix = 0$. $(A - 2I)x = 0$.

$A - 2I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.

$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{\text{swap } (1) \leftrightarrow (2)} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right) \xrightarrow{\text{add } (1) \text{ to } (2)} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$.

let $x_2 = t$
$x_1 - t = 0 \Rightarrow x_1 = t$
$x_2 = t$.
The eigenspace is $\left\{\begin{pmatrix} t \\ t \end{pmatrix} : t \in \mathbb{R}\right\}$.

To find a basis for the eigenspace, take each parameter in turn: let it equal 1, all others $= 0$.

Basis: $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.

$\lambda = 0$: $(A - 0I)x = 0$.

$A - 0I = A$, $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow[\text{to } (2)]{\text{add } -(1)} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

let $x_2 = t$, $x_1 = -t$. The eigenspace is $\left\{\begin{pmatrix} -t \\ t \end{pmatrix} : t \in \mathbb{R}\right\}$

Basis: $\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$.

If A has eigenvalue λ, to find the eigenvectors, solve: $(A - \lambda I)x = 0$. To find a basis for each eigenspace, let each parameter in turn be 1, and all others 0.

e.g. let $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Its eigenvalues are $-1, 3$.

$\lambda = -1$: $A - (-1)I = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \xrightarrow[\text{by } 1/2]{\text{mult } (1)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \xrightarrow[\text{to } (2)]{\text{add } -2(1)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \xrightarrow[(2)(3)]{\text{swap}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{by } 1/4]{\text{mult } (2)}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

let $x_2 = t$, $x_1 = -t$, $x_3 = 0$.

Eigenspace: $\left\{\begin{pmatrix} -t \\ t \\ 0 \end{pmatrix} : t \in \mathbb{R}\right\}$. Basis: $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$

$\lambda = 3$: $A - 3I = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{by } -\frac{1}{2}]{\text{mult } (1)} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{to } (2)]{\text{add } -2(1)} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Let $x_2 = t$, $x_3 = u$, then $x_1 = t$.

Eigenspace: $\left\{\begin{pmatrix} t \\ t \\ u \end{pmatrix} : t, u \in \mathbb{R}\right\}$. Basis: $\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$.

- last time — Cramer's rule.

- eigenvalues, eigenvectors, eigenspaces.

$Ax = \lambda x$.

- given x, calculate Ax, see if it is a scalar multiple of x.

- given λ: solve $\underline{(A-\lambda I)x=0}$

- find a basis for eigenspace.

λ is an eigenvalue for A if and only if

$(A-\lambda I)x = 0$ has a nontrivial solution.

This happens if and only if $A-\lambda I$ is not invertible.

But this ~~can~~ occurs if and only if $\det(A-\lambda I) = 0$.

We call $\det(A-\lambda I)$ the <u>characteristic polynomial</u> of A.

We call $\det(A-\lambda I)=0$ the <u>characteristic equation</u>.

To find eigenvalues, we solve the characteristic equation.

We then know how to find eigenvectors

e.g. $A=\begin{pmatrix}1&5\\5&1\end{pmatrix}$. Find eigenvalues, eigenvectors.

$$0=\det(A-\lambda I)=\det\left(\begin{pmatrix}1&5\\5&1\end{pmatrix}-\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}\right)=\det\begin{pmatrix}1-\lambda&5\\5&1-\lambda\end{pmatrix}$$

$$=(1-\lambda)^2-25=\lambda^2-2\lambda-24$$

$$=(\lambda-6)(\lambda+4),\text{ so } \lambda=6,-4.$$

$\lambda=6$: $A-6I=\begin{pmatrix}-5&5\\5&-5\end{pmatrix}$.

$$\left(\begin{array}{cc|c}-5&5&0\\5&-5&0\end{array}\right)\xrightarrow[\text{by }-1/5]{\text{mult ①}}\left(\begin{array}{cc|c}1&-1&0\\5&-5&0\end{array}\right)\xrightarrow[\text{to ②}]{\text{add }-5\text{①}}\left(\begin{array}{cc|c}1&-1&0\\0&0&0\end{array}\right).$$

let $x_2=t$
$x_1=t$.

Basis for eigenspace: $\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$.

$\lambda=-4$: $A-(-4)I=\begin{pmatrix}5&5\\5&5\end{pmatrix}$

$$\left(\begin{array}{cc|c}5&5&0\\5&5&0\end{array}\right)\xrightarrow[\text{by }1/5]{\text{mult ①}}\left(\begin{array}{cc|c}1&1&0\\5&5&0\end{array}\right)\xrightarrow[\text{to ②}]{\text{add }-5\text{①}}\left(\begin{array}{cc|c}1&1&0\\0&0&0\end{array}\right).$$

let $x_2=t$
$x_1=-t$

Basis: $\left\{\begin{pmatrix}-1\\1\end{pmatrix}\right\}$.

e.g. $A=\begin{pmatrix}1&4\\-3&1\end{pmatrix}$, $0=\det\begin{pmatrix}1-\lambda&4\\-3&1-\lambda\end{pmatrix}=(1-\lambda)^2+12\geq 12$.
No real eigenvalues.

e.g. $A=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$.

$$0=\det\begin{pmatrix}1-\lambda&1&1\\1&1-\lambda&1\\1&1&1-\lambda\end{pmatrix}=(1-\lambda)\det\begin{pmatrix}1-\lambda&1\\1&1-\lambda\end{pmatrix}-\det\begin{pmatrix}1&1\\1&1-\lambda\end{pmatrix}+\det\begin{pmatrix}1&1-\lambda\\1&1\end{pmatrix}$$

$$=(1-\lambda)\left((1-\lambda)^2-1\right)-(-\lambda)+(\lambda)$$

$$=(1-\lambda)(\lambda^2-2\lambda)+2\lambda$$

$$= -\lambda^3 + 3\lambda^2 = \lambda^2(-\lambda+3).$$

$$\lambda = 0, 3.$$

$\lambda = 0$: $A - 0I = A$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ \boxed{1} & 1 & 1 & 0 \\ \boxed{1} & 1 & 1 & 0 \end{array}\right) \begin{array}{c} \text{add } -1 \text{ (1) to (2)} \\ \text{add } -1 \text{ (1) to (3)} \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Let $x_2 = t$, $x_3 = u$. Then $x_1 = -t-u$. Basis $= \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$\lambda = 3$: $A - 3I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \xrightarrow[\text{(1), (2)}]{\text{swap}} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ \boxed{-2} & 1 & 1 & 0 \\ \boxed{1} & 1 & -2 & 0 \end{array}\right) \begin{array}{c} \text{add } 2 \text{ (1) to (2)} \\ \text{add } -1 \text{ (1) to (3)} \end{array} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \begin{array}{c} \text{mult (2)} \\ \text{by } -\frac{1}{3} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & \boxed{-2} & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & \boxed{3} & -3 & 0 \end{array}\right) \begin{array}{c} \text{add } 2 \text{ (2) to (1)} \\ \text{add } -3 \text{ (2) to (3)} \end{array} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \quad \begin{array}{l} \text{Let } x_3 = t. \\ x_1 = t \\ x_2 = t. \end{array} \quad \text{Basis} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

If A is upper/lower triangular, the eigenvalues are the diagonal entries.

e.g. $A = \begin{pmatrix} 3 & 4 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 7 \end{pmatrix}$. $0 = \det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 4 & 5 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 7-\lambda \end{pmatrix}$

$$= (3-\lambda)(2-\lambda)(7-\lambda).$$

Let A be a square matrix, n a positive integer. Then

$$A^n = \underbrace{AA\cdots A}_{n \text{ times}}.$$

e.g. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$.

$A^m A^n = A^{m+n}$

$(A^m)^n = A^{mn}$.

If we have a polynomial $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$

then $f(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n$.

CAYLEY-HAMILTON THM: A satisfies its characteristic equation.

e.g. $A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$, char poly: $\lambda^2 - 2\lambda - 24$.

$\longrightarrow$ $A^2 - 2A - 24I = 0$.

$$\begin{pmatrix} 26 & 10 \\ 10 & 26 \end{pmatrix} - 2\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix} - 24\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We can use this to calculate higher powers of matrices.

If A is $n \times n$ then its characteristic poly has degree n.

We can write A^n as a linear combination of $I, A, A^2, \ldots, A^{n-1}$.

(e.g. above $A^2 = 2A + 24I$).

This allows us to write all powers of A in terms of

$I, A, A^2, \ldots, A^{n-1}$.

(e.g. also $A^2 = 2A + 24I$

$$A^3 = A^2A = 2A^2 + 24A$$

$$= 2(2A + 24I) + 24A$$

$$= 28A + 48I.)$$

If $A^m = c_0 I + c_1 A + c_2 A^2 + \cdots + c_{n-1} A^{n-1}$

then for every eigenvalue λ of A,

$\lambda^m = c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_{n-1}\lambda^{n-1}$.

If we have n different eigenvalues, we can obtain a system of n equations in n unknowns, and solve for $c_0, c_1, \ldots, c_{n-1}$.

- last time - characteristic polynomial $\det(A-\lambda I)$

  - " equation $\det(A-\lambda I)=0$

  - eigenvalues satisfy this equation

  - A triangular - eigenvalues are diagonal entries.

  - A^m, polynomials in terms of A

  - Cayley-Hamilton Thm.

  - if A is $n\times n$, $A^m = c_0 I + c_1 A + c_2 A^2 + \cdots + c_{n-1} A^{n-1}$.

  - the eigenvalues satisfying the same equation.

- e.g. $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. Find A^{10}.

  As A is 2×2, $\underline{A^{10} = c_0 I + c_1 A}$.

  As A is upper triangular, the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2$.

  $c_0 + c_1 \lambda = \lambda^{10}$.

  $$\begin{aligned} c_0 + c_1 &= 1 \\ c_0 + 2c_1 &= 1024 \end{aligned}$$

  $c_1 = 1023$, $c_0 = -1022$.

  $A^{10} = -1022\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 1023\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$

-e.g. $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, find A^6.

~~[illegible]~~ $c_0 I + c_1 A = A^6$.

$0 = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$

$= (\lambda - 3)(\lambda + 1)$.

$\lambda = 3, -1$.

$c_0 + c_1 \lambda = \lambda^6$

$$\begin{aligned} c_0 + 3c_1 &= 729 \\ c_0 - c_1 &= 1 \\ \hline 4c_1 &= 728, \quad c_1 = 182, \quad c_0 = 183. \end{aligned}$$

$A^6 = 183 I + 182 A$.

A matrix A is symmetric if $A = A^T$.

-e.g. $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & -7 \end{pmatrix}$.

THM: If A is a symmetric matrix, all of its eigenvalues are real!

THM: If A is symmetric, and it has two different eigenvalues λ and μ, then if $Ax = \lambda x$ and $Ay = \mu y$, then x and y are orthogonal.

PF: $y^TAx = y^T\lambda x = \lambda y^T x = \lambda(x\cdot y)$. $\quad (3\ 5\ 7)\begin{pmatrix}1\\2\\-6\end{pmatrix}$

$\parallel$

$y^TAx = y^TA^Tx = (Ay)^Tx = (\mu y)^Tx = \mu(x\cdot y)$.

$\lambda(x\cdot y) = \mu(x\cdot y)$. So $x\cdot y = 0$ (yay) or $\lambda = \mu$ $\cancel{\cancel{\ }}$.

-e.g. $A = \begin{pmatrix}1&2&0\\2&1&0\\0&0&3\end{pmatrix}$. Let $x = \begin{pmatrix}1\\1\\1\end{pmatrix}$. $Ax = \begin{pmatrix}3\\3\\3\end{pmatrix} = 3x$.

$y = \begin{pmatrix}-1\\1\\0\end{pmatrix}$. $Ay = \begin{pmatrix}-1\\1\\0\end{pmatrix} = (-1)y$.

We see that $x\cdot y = 0$.

An $n\times n$ matrix A is orthogonal if it is invertible, and $A^{-1} = A^T$.

-e.g. $A = \begin{pmatrix}1/\sqrt{2}&-1/\sqrt{2}&0\\1/\sqrt{2}&1/\sqrt{2}&0\\0&0&1\end{pmatrix}$, $AA^T = \begin{pmatrix}1/\sqrt{2}&-1/\sqrt{2}&0\\1/\sqrt{2}&1/\sqrt{2}&0\\0&0&1\end{pmatrix}\begin{pmatrix}1/\sqrt{2}&1/\sqrt{2}&0\\-1/\sqrt{2}&1/\sqrt{2}&0\\0&0&1\end{pmatrix}$

$= \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

THM: An $n\times n$ matrix A is orthogonal if and only if its columns form an orthonormal set

Suppose $A = (x_1\ x_2 \cdots x_n)$. Then

$A^TA = \begin{pmatrix}x_1^T\\x_2^T\\\vdots\\x_n^T\end{pmatrix}(x_1 \cdots x_n) = \begin{pmatrix}x_1^Tx_1 & x_1^Tx_2 & \cdots\\ x_2^Tx_1 & x_2^Tx_2 & \cdots\end{pmatrix}$.

For this to be I, $\underset{\substack{\parallel\\ x_i\cdot x_j}}{x_i^Tx_j} = \begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$.

Let A be a square matrix. Then A is said to be diagonalizable if there is an invertible matrix P so that

$P^{-1}AP = D$, for some diagonal matrix D.

Thm: If A is $n \times n$, then A is diagonalizable $\iff$ it has n linearly independent eigenvectors.

Suppose we have $Ax_i = \lambda_i x_i$. Let $P = (x_1 x_2 \cdots x_n)$.

$AP = (Ax_1 Ax_2 \cdots Ax_n) = (\lambda_1 x_1 \; \lambda_2 x_2 \cdots \lambda_n x_n)$.

Let $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{pmatrix}$.

$PD = (x_1 \cdots x_n)\begin{pmatrix} \lambda_1 & & 0 \\ 0 & & \lambda_n \end{pmatrix} = (\lambda_1 x_1 \cdots \lambda_n x_n)$.

$AP = PD$. Now, P is invertible if and only if its columns are linearly independent.

Given an $n \times n$ matrix A, we find the eigenvalues, and a basis for each eigenspace. If n vectors were obtained, A is diagonalizable.

Let P = the matrix with columns equal to these basis vectors.

Then $P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ 0 & & \lambda_n \end{pmatrix}$ where the λ_i are the eigenvalues in the order in which they were used.

If we get fewer than n basis vectors, A is not diagonalizable.

COROLLARY: If A is $n\times n$ and has n different eigenvalues, it is diagonalizable.

- e.g. $A=\begin{pmatrix}1&2\\2&1\end{pmatrix}$. $0=\det(A-\lambda I)=\det\begin{pmatrix}1-\lambda&2\\2&1-\lambda\end{pmatrix}=(1-\lambda)^2-4$

$=\lambda^2-2\lambda-3$

$\lambda=3,-1.$ $\qquad =(\lambda-3)(\lambda+1)$

$\lambda=3$: $A-3I=\begin{pmatrix}-2&2\\2&-2\end{pmatrix}$.

$\left(\begin{array}{cc|c}-2&2&0\\2&-2&0\end{array}\right) \xrightarrow[\text{by } -1/2]{\text{mult } ①} \left(\begin{array}{cc|c}1&-1&0\\2&-2&0\end{array}\right) \xrightarrow[\text{to } ②]{\text{add } -2①} \left(\begin{array}{cc|c}1&-1&0\\0&0&0\end{array}\right)$.

Let $x_2=t$, $x_1=t$. Basis: $\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$.

$\lambda=-1$: $A-(-1)I=\begin{pmatrix}2&2\\2&2\end{pmatrix}$.

$\left(\begin{array}{cc|c}2&2&0\\2&2&0\end{array}\right) \xrightarrow[\text{by } 1/2]{\text{mult } ①} \left(\begin{array}{cc|c}1&1&0\\2&2&0\end{array}\right) \xrightarrow[\text{to } ②]{\text{add } -2①} \left(\begin{array}{cc|c}1&1&0\\0&0&0\end{array}\right)$

Let $x_2=t$, $x_1=-t$. Basis: $\left\{\begin{pmatrix}-1\\1\end{pmatrix}\right\}$.

$P=\begin{pmatrix}1&-1\\1&1\end{pmatrix}$, $P^{-1}AP=D=\begin{pmatrix}3&0\\0&-1\end{pmatrix}$.

- last time — calculating powers of a matrix.

  - symmetric matrix.

  - symmetric $\Rightarrow$ all eigenvalues real

    $\Rightarrow$ eigenvectors corresponding to different eigenvalues are orthogonal.

- orthogonal $\Leftrightarrow$ orthonormal columns.

- diagonalizable. $P^{-1}AP = D$, D diagonal.

- $n \times n$ A diagonalizable $\Leftrightarrow$ n linearly indep. eigenvectors.

  If so, let P have these eigenvectors as its columns.

  Then P is invertible, $P^{-1}AP = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$

  d_i eigenvalues.

If A is $n \times n$, n different eigenvalues, then it is diagonalizable.

If there are fewer than n different eigenvalues, ???

- e.g. $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$. Eigenvalue: 1.

$\lambda = 1$: $A - 1I = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$.

$\left(\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{by } 1/3]{\text{mult } (1)} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$. Let $x_1 = t$, $x_2 = 0$. Basis: $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

Not enough vectors, not diagonalizable.

- e.g. $A = \begin{pmatrix} 1 & 3 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. $0 = \det \begin{pmatrix} 1-\lambda & 3 & 0 \\ -1 & 5-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda) \det \begin{pmatrix} 1-\lambda & 3 \\ -1 & 5-\lambda \end{pmatrix}$

$= (2-\lambda)(\lambda^2 - 6\lambda + 8)$

$= (2-\lambda)(\lambda - 2)(\lambda - 4)$.

$\lambda = 2, 4$.

$\lambda = 2$: $A - 2I = \begin{pmatrix} -1 & 3 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$\left(\begin{array}{ccc|c} -1 & 3 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{by } -1]{\text{mult } (1)} \left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{to } (2)]{\text{add } (1)} \left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

Let $x_2 = t$, $x_3 = u$, $x_1 = 3t$. Basis: $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$\lambda = 4$: $A - 4I = \begin{pmatrix} -3 & 3 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

$\left(\begin{array}{ccc|c} -3 & 3 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \xrightarrow[\text{by } -1/3]{\text{mult } (1)} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \xrightarrow[\text{to } (2)]{\text{add } (1)} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \xrightarrow{\text{swap } (2), (3)}$

$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{by } -\frac{1}{2}]{\text{mult } (2)} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. Let $x_2 = t$, $x_1 = t$, $x_3 = 0$.

Basis: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

A is diagonalizable. Let $P = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

$P^{-1} A P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

An $n \times n$ matrix A is orthogonally diagonalizable if there exists an orthogonal matrix P so that $P^{-1}AP$ is diagonal.

$$P^{-1}AP = P^TAP.$$

A is orthogonally diagonalizable if and only if it is symmetric.

We need an orthonormal basis for each eigenspace.

e.g. $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$. $0 = \det\begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda - 8$

$= (\lambda - 4)(\lambda + 2)$.

$\lambda = 4, -2$.

$\lambda = 4$: $A - 4I = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}$.

$\left(\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right) \xrightarrow[R_1 \cdot -1/3]{\text{mult } (1)} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right) \xrightarrow[\text{to } (2)]{\text{add } -3(1)} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$.

Let $x_2 = t$, $x_1 = t$. Basis: $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$. Normalize: $\left\{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right\}$.

$\lambda = -2$: $A + 2I = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$

$\left(\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right) \xrightarrow[R_1 \cdot 1/3]{\text{mult } (1)} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 3 & 3 & 0 \end{array}\right) \xrightarrow[\text{to } (2)]{\text{add } -3(1)} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$.

Let $x_2 = t$, $x_1 = -t$. Basis: $\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$. Normalize: $\left\{\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right\}$.

$P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$, $P^{-1}AP = P^TAP = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$.

e.g. $A=\begin{pmatrix}3&3&3\\3&3&3\\3&3&3\end{pmatrix}$. $0=\det\begin{pmatrix}3-\lambda&3&3\\3&3-\lambda&3\\3&3&3-\lambda\end{pmatrix}=(3-\lambda)\det\begin{pmatrix}3-\lambda&3\\3&3-\lambda\end{pmatrix}-3\det\begin{pmatrix}3&3\\3&3-\lambda\end{pmatrix}+3\det\begin{pmatrix}3&3-\lambda\\3&3\end{pmatrix}$

$0=(3-\lambda)(\lambda^2-6\lambda)-3(-3\lambda)+3(3\lambda)$

$=-\lambda^3+9\lambda^2=\lambda^2(-\lambda+9),\ \lambda=0,9$

$\lambda=0$: $A-0I=A$.

$\left(\begin{array}{ccc|c}3&3&3&0\\3&3&3&0\\3&3&3&0\end{array}\right)\xrightarrow[\frac{1}{3}]{\text{mult (1)}}\left(\begin{array}{ccc|c}1&1&1&0\\3&3&3&0\\3&3&3&0\end{array}\right)\begin{array}{l}\text{add }-3(1)\text{ to }(2)\\\text{add }-3(1)\text{ to }(3)\end{array}\left(\begin{array}{ccc|c}1&1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right).$

Let $x_2=t$, $x_3=u$. Basis: $\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$.
$x_1=-t-u$.

Apply Gram-Schmidt. $\vec{u}_1=\begin{pmatrix}-1\\1\\0\end{pmatrix}$, $\vec{u}_2=\begin{pmatrix}-1\\0\\1\end{pmatrix}$.

$\vec{v}_1=\vec{u}_1=\begin{pmatrix}-1\\1\\0\end{pmatrix}$, $\vec{v}_2=\vec{u}_2-\frac{\vec{u}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1$

$=\begin{pmatrix}-1\\0\\1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix}.$

Double $\vec{v}_2$, $\vec{v}_2=2\begin{pmatrix}-1\\0\\1\end{pmatrix}-\begin{pmatrix}-1\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\-1\\2\end{pmatrix}.$

Normalize: $\left\{\begin{pmatrix}-1/\sqrt2\\1/\sqrt2\\0\end{pmatrix},\begin{pmatrix}-1/\sqrt6\\-1/\sqrt6\\2/\sqrt6\end{pmatrix}\right\}$.

$\lambda=9$: $A-9I=\begin{pmatrix}-6&3&3\\3&-6&3\\3&3&-6\end{pmatrix}$.

$\left(\begin{array}{ccc|c}-6&3&3&0\\3&-6&3&0\\3&3&-6&0\end{array}\right)\xrightarrow[(1),(2)]{\text{swap}}\left(\begin{array}{ccc|c}3&-6&3&0\\-6&3&3&0\\3&3&-6&0\end{array}\right)\xrightarrow[\frac{1}{3}]{\text{mult (1)}}\left(\begin{array}{ccc|c}1&-2&1&0\\-6&3&3&0\\3&3&-6&0\end{array}\right)\begin{array}{l}\text{add }6(1)\text{ to }(2)\\\text{add }-3(1)\text{ to }(3)\end{array}$

$\left(\begin{array}{ccc|c}1&-2&1&0\\0&-9&9&0\\0&9&-9&0\end{array}\right)\xrightarrow[-\frac{1}{9}]{\text{mult (2)}}\left(\begin{array}{ccc|c}1&-2&1&0\\0&1&-1&0\\0&9&-9&0\end{array}\right)\begin{array}{l}\text{add }2(2)\text{ to }(1)\\\text{add }-9(2)\text{ to }(3)\end{array}\left(\begin{array}{ccc|c}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{array}\right).$

Let $x_3 = t$ $x_1 = t$
$x_2 = t$. Basis: $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$. Normalize: $\left\{\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}\right\}$.

Let $P = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$. $P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{pmatrix}$.

Cryptography: send a secret message

Select a key, an $n \times n$ matrix A so that

$\det(A) = \pm 1$, so that A and A^{-1} have integer entries.

We will then convert our message to numbers:

$A = 1, B = 2, \ldots, Z = 26, \_ = 0$. We will arrange

our message in an $n \times k$ matrix, where k is determined

by the message size. (Pad out any blanks at the end with zeros).

Suppose our message is M. To encrypt, calculate

$B = AM$, and send B. To decrypt, calculate

$A^{-1}B = A^{-1}AM = IM = M$.

e.g. $A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$. XENA_RULES.

24, 5, 14, 1, 0, 18, 21, 12, 5, 19.

$M = \begin{pmatrix} 24 & 5 & 14 & 1 & 0 \\ 18 & 21 & 12 & 5 & 19 \end{pmatrix}$.

$$B = AM = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 24 & 5 & 14 & 1 & 0 \\ 18 & 21 & 12 & 5 & 19 \end{pmatrix}$$

$$= \begin{pmatrix} 102 & 73 & 64 & 17 & 57 \\ 162 & 120 & 102 & 28 & 95 \end{pmatrix} \leftarrow \text{send.}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$$

$$M = A^{-1}B = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 102 & 73 & 64 & 17 & 57 \\ 162 & 120 & 102 & 28 & 95 \end{pmatrix}$$

$$= \begin{pmatrix} 24 & 5 & 14 & 1 & 0 \\ 18 & 21 & 12 & 5 & 19 \end{pmatrix}.$$

XENA -
RULES.

- last time - diagonalizing.

 - orthogonally diagonalizable $\Leftrightarrow$ symmetric.

 - cryptography. $\det(A)=\pm 1$, $M \to AM \to A^{-1}AM = M$.

Mod 2 arithmetic

$\cdot$	0	1
0	0	0
1	0	1

+	0	1
0	0	1
1	1	0

- e.g. $1+1+0+1 = 0+0+1 = 0+1 = 1$.

A binary string is a list of zeroes and ones.

In a parity check code, we append a check bit to our string, being the mod 2 sum of the numbers in the string.

- e.g. (110101) $\quad 1+1+0+1+0+1=0$.

We send (1101010). The recipient checks that the mod 2 sum is 0. If so, the last ~~dd~~ bit is deleted to obtain the message. If not, there is an error

- e.g. (101101) ✓ Message: (10110)

(101100) error.

Advantages: easy, efficient

Disadvantages: can't fix errors.

two errors? eek!

<u>Hamming code</u>: Our message will be a binary string

of length 4, (w_1, w_2, w_3, w_4).

We have three check bits: $c_1 = w_1 + w_2 + w_4$.

$$\left.\begin{aligned} c_1 &= w_1 + w_2 + w_4 \\ c_2 &= w_1 + w_3 + w_4 \\ c_3 &= w_2 + w_3 + w_4 \end{aligned}\right\} \bmod 2.$$

Our encoded message is: $(c_1 c_2 w_1 c_3 w_2 w_3 w_4)$.

- e.g. $(1\,0\,0\,1)$ $\quad c_1 = 1+0+1 = 0,\ c_2 = 1+0+1 = 0,\ c_3 = 0+0+1 = 1.$

we get $(0\,0\,1\,1\,0\,0\,1)$.

To check the message C, the recipient forms the Hamming

matrix: $H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$

The recipient calculates HC^T

$$\begin{pmatrix} 0&0&0&1&1&1&1\\ 0&1&1&0&0&1&1\\ 1&0&1&0&1&0&1 \end{pmatrix}\begin{pmatrix} c_1\\ c_2\\ w_1\\ c_3\\ w_2\\ w_3\\ w_4 \end{pmatrix} = \begin{pmatrix} c_3+w_2+w_3+w_4\\ c_2+w_1+w_3+w_4\\ c_1+w_1+w_2+w_4 \end{pmatrix} = \begin{pmatrix} 0\\0\\0 \end{pmatrix}.$$

If we get $\begin{pmatrix}0\\0\\0\end{pmatrix}$, our message is correct, and we drop the check bits, to get the original message.

- e.g. we receive (1010101)

$$\begin{pmatrix} 0&0&0&1&1&1&1\\ 0&1&1&0&0&1&1\\ 1&0&1&0&1&0&1 \end{pmatrix}\begin{pmatrix} 1\\0\\1\\0\\1\\0\\1 \end{pmatrix} = \begin{pmatrix} 0\\0\\0 \end{pmatrix}.$$

Valid. Original message (1101).

Suppose one bit gets flipped. Our message should have been C, but instead it is $C+E$, where

$$E = (0\cdots 0\underset{i}{1}0\cdots 0)$$

$$H(C+E)^T = HC^T + HE^T = HE^T = \text{column } i \text{ of } H.$$

- e.g. we receive (1010111).

$$\begin{pmatrix} 0&0&0&1&1&1&1\\ 0&1&1&0&0&1&1\\ 1&0&1&0&1&0&1 \end{pmatrix}\begin{pmatrix} 1\\0\\1\\0\\1\\1\\1 \end{pmatrix} = \begin{pmatrix} 1\\1\\0 \end{pmatrix}, \text{ column 6 of } H.$$

Error in bit 6, message should have been (1010101), so the original message (1101).

Advantages: can fix any 1 error.

will know there is a problem with 2 errors.

Disadvantages: less efficient, more complicated

Read Chapter 2.

Chapter 3 - Probability (3.1-3.7)

An experiment is any procedure leading to an outcome

The sample space, S, is the set of all possible outcomes

for an experiment.

An event is any collection of outcomes; that is, it is

a subset of the sample space.

- e.g. flip a coin 3 times, record the results.

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

- e.g. flip a coin 3 times, count the heads

$S = \{0, 1, 2, 3\}$

- e.g. roll two dice, record the total

$S = \{2, 3, \ldots, 12\}$.

- e.g. roll 2 dice, record the results.

$S = \{11, 12, \ldots, 16, 21, 22, \ldots, 66\}$.

In <u>discrete probability</u> (chapters 3+4), we have either finitely many or countably many possible outcomes.

A <u>countable</u> set has terms that can be put in an infinite sequence.

- e.g. the positive integers: $1, 2, 3, 4, \ldots$.
- e.g. the integers: $0, 1, -1, 2, -2, 3, -3, \ldots$
- e.g. the rational numbers are countable.
- e.g. the real numbers are uncountable.
- e.g. $[0,1)$ is uncountable.

In <u>continuous probability</u> (chapter 5+), the possible outcomes are either all real numbers, or an interval.

- ~~the~~ e.g. flip a coin until heads appears, count the flips

$S = \{1, 2, 3, \ldots\}$.

Let E and F be events. Then the intersection of E and F, $E \cap F$, is the event that E and F both occur. It is the set of all outcomes lying in E and F simultaneously.

- roll a die, $E = \{1, 3, 5\}$, $F = \{3, 6\}$, $G = \{2, 4, 6\}$.

$E \cap F = \{3\}$, $F \cap G = \{6\}$

$E \cap G = \emptyset$ "empty set".

We say that E and G are mutually exclusive.

$E \cap E = E$, $E \cap \emptyset = \emptyset$, $E \cap S = E$

e.g. solve $-x_1+3x_2+4x_3=5$
$2x_1-7x_2+3x_3=2$.

$$\begin{pmatrix}-1&3&4&\big|&5\\2&-7&3&\big|&2\end{pmatrix}\xrightarrow[\text{by } -1]{\text{mult } ①}\begin{pmatrix}1&-3&-4&\big|&-5\\\boxed{2}&-7&3&\big|&2\end{pmatrix}\xrightarrow[\text{to } ②]{\text{add } -2①}\begin{pmatrix}1&-3&-4&\big|&-5\\0&-1&11&\big|&12\end{pmatrix}\xrightarrow[\text{by } -1]{\text{mult } ②}$$

$$\begin{pmatrix}1&\boxed{-3}&-4&\big|&-5\\0&1&-11&\big|&-12\end{pmatrix}\xrightarrow[\text{to } ①]{\text{add } 3②}\begin{pmatrix}1&0&-37&\big|&-41\\0&1&-11&\big|&-12\end{pmatrix}.$$

let $x_3=t$, $x_1=-41+37t$, $x_2=-12+11t$.

e.g. find the equation of the plane passing through $(1,2,1)$, $(2,3,4)$, $(3,1,8)$.

vectors in the plane: $\vec{a}=\langle 1,1,3\rangle$, $\vec{b}=\langle 2,-1,7\rangle$.

$\vec{n}=\vec{a}\times\vec{b}=\det\begin{pmatrix}\vec{i}&\vec{j}&\vec{k}\\1&1&3\\2&-1&7\end{pmatrix}=\begin{vmatrix}1&3\\-1&7\end{vmatrix}\vec{i}-\begin{vmatrix}1&3\\2&7\end{vmatrix}\vec{j}+\begin{vmatrix}1&1\\2&-1\end{vmatrix}\vec{k}$

$=\langle 10,-1,-3\rangle$.

$10x-y-3z=d$, $10(1)-2-3(1)=d$, so $d=5$.

$10x-y-3z=5$.

e.g. let $A=\begin{pmatrix}0&1&3\\2&9&12\\1&3&2\end{pmatrix}$. Is A invertible? If so, find A^{-1}.

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 3 & 1 & 0 & 0 \\ 2 & 9 & 12 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[(1),(3)]{\text{swap}} \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 0 & 1 \\ \boxed{2} & 9 & 12 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \end{array}\right) \xrightarrow[\text{to } (2)]{\text{add } -2(1)} \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 0 & 1 \\ 0 & 3 & 8 & 0 & 1 & -2 \\ 0 & 1 & 3 & 1 & 0 & 0 \end{array}\right) \xrightarrow[(2),(3)]{\text{swap}}$$

$$\left(\begin{array}{ccc|ccc} 1 & \boxed{3} & 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & \boxed{3} & 8 & 0 & 1 & -2 \end{array}\right) \xrightarrow[\text{add } -3(2) \text{ to } (3)]{\text{add } -3(2) \text{ to } (1)} \left(\begin{array}{ccc|ccc} 1 & 0 & -7 & -3 & 0 & 1 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -3 & 1 & -2 \end{array}\right) \xrightarrow[\text{by } -1]{\text{mult } (3)}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \boxed{-7} & -3 & 0 & 1 \\ 0 & 1 & \boxed{3} & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 2 \end{array}\right) \xrightarrow[\text{add } -3(3) \text{ to } (2)]{\text{add } 7(3) \text{ to } (1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 18 & -7 & 15 \\ 0 & 1 & 0 & -8 & 3 & -6 \\ 0 & 0 & 1 & 3 & -1 & 2 \end{array}\right)$$

$$\boxed{\begin{matrix} 18 & -7 & 15 \\ -8 & 3 & -6 \\ 3 & -1 & 2 \end{matrix}} = A^{-1}$$

- e.g. find all 3 forms of the line through $(1,-3,4)$, $(3,1,7)$.

vector: $\langle x,y,z\rangle = \langle 1,-3,4\rangle + t\langle 2,-4,3\rangle$.

parametric: $x = 1+2t$, $y = 3-4t$, $z = 4+3t$.

symmetric: $\dfrac{x-1}{2} = \dfrac{y+3}{-4} = \dfrac{z-4}{3}$.

- e.g. are $\langle 1,3,2\rangle, \langle 2,5,1\rangle, \langle 3,8,3\rangle$ linearly dep or indep?

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 3 & 8 & 3 \end{pmatrix} \xrightarrow[\text{add } -3(1) \text{ to } (3)]{\text{add } -2(1) \text{ to } (2)} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & -1 & -3 \end{pmatrix} \xrightarrow[\text{to } (3)]{\text{add } -(2)} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Zero row, so linearly dependent.

rank = 2.

e.g. $V = \{ \langle 2a-b, a, b \rangle : a, b \in \mathbb{R} \}$.

(i) Is $\langle 1, 3, 7 \rangle \in V$?

(ii) Is V a subspace of $\mathbb{R}^3$?

(i) Can we solve
$$\begin{aligned} 2a-b &= 1 \\ a &= 3 \\ b &= 7. \end{aligned}$$

If $a=3$, $b=7$, then $2a-\cancel{b} = 2(3)-7 = -1$, not 1.

$\langle 1, 3, 7 \rangle \notin V$.

(ii) Let $a=b=0$, $\langle 0,0,0 \rangle \in V$.

$\langle 2a_1-b_1, a_1, b_1 \rangle + \langle 2a_2-b_2, a_2, b_2 \rangle$

$= \langle 2a_1-b_1+2a_2-b_2, a_1+a_2, b_1+b_2 \rangle$

$= \langle 2(a_1+a_2)-(b_1+b_2), a_1+a_2, b_1+b_2 \rangle \in V$.

$\lambda \langle 2a-b, a, b \rangle = \langle 2\lambda a-\lambda b, \lambda a, \lambda b \rangle \in V$.

V is a subspace.

$v \in V$; $(-1)v \in V$, so $-v \in V$, and so $v+(-v) = 0 \in V$.

-e.g. $A=\begin{pmatrix}2&0&0\\0&1&2\\0&-1&4\end{pmatrix}$. (i) Is it diagonalizable? (ii) Is it orthogonally diagonalizable?

(ii) No, not symmetric.

$$\text{(i) } 0=\det\begin{pmatrix}2-\lambda&0&0\\0&1-\lambda&2\\0&-1&4-\lambda\end{pmatrix}=(2-\lambda)\det\begin{pmatrix}1-\lambda&2\\-1&4-\lambda\end{pmatrix}$$

$$=(2-\lambda)(\lambda^2-5\lambda+6)$$

$$=(2-\lambda)(\lambda-2)(\lambda-3)$$

$$\lambda=2,3.$$

$\lambda=2$: $A-2I=\begin{pmatrix}0&0&0\\0&-1&2\\0&-1&2\end{pmatrix}$.

$$\left(\begin{array}{ccc|c}0&0&0&0\\0&-1&2&0\\0&-1&2&0\end{array}\right)\xrightarrow[\text{①,②}]{\text{swap}}\left(\begin{array}{ccc|c}0&-1&2&0\\0&0&0&0\\0&-1&2&0\end{array}\right)\xrightarrow[\text{by }-1]{\text{mult ①}}\left(\begin{array}{ccc|c}0&1&-2&0\\0&0&0&0\\0&-1&2&0\end{array}\right)\xrightarrow[\text{to ③}]{\text{add ①}}\left(\begin{array}{ccc|c}0&1&-2&0\\0&0&0&0\\0&0&0&0\end{array}\right)$$

Let $x_1=t$, $x_3=u$, $x_2=2u$. $B_{(i)}=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\1\end{pmatrix}\right\}$.

$\lambda=3$: $A-3I=\begin{pmatrix}-1&0&0\\0&-2&2\\0&-1&1\end{pmatrix}$.

$$\left(\begin{array}{ccc|c}-1&0&0&0\\0&-2&2&0\\0&-1&1&0\end{array}\right)\xrightarrow[\text{by }-1]{\text{mult ①}}\left(\begin{array}{ccc|c}1&0&0&0\\0&-2&2&0\\0&-1&1&0\end{array}\right)\xrightarrow[\text{by }-\frac{1}{2}]{\text{mult ②}}\left(\begin{array}{ccc|c}1&0&0&0\\0&1&-1&0\\0&-1&1&0\end{array}\right)\xrightarrow[\text{to ③}]{\text{add ②}}\left(\begin{array}{ccc|c}1&0&0&0\\0&1&-1&0\\0&0&0&0\end{array}\right)$$

Let $x_3=t$. $x_1=0$, $x_2=t$. $B_{(ii)}=\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$.

$P=\begin{pmatrix}1&0&0\\0&2&1\\0&1&1\end{pmatrix}$. $P^{-1}AP=\begin{pmatrix}2&0&0\\0&2&0\\0&0&3\end{pmatrix}$.

- e.g. encode and decode the message HOWDY using the key $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$

$M = \begin{pmatrix} 8 & 15 & 23 \\ 4 & 25 & 0 \end{pmatrix}$

encode: $B = AM = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 8 & 15 & 23 \\ 4 & 25 & 0 \end{pmatrix} = \begin{pmatrix} 20 & 90 & 23 \\ 36 & 155 & 46 \end{pmatrix}$.

$A^{-1} = \frac{1}{-1}\begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$.

check $A^{-1}B = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 20 & 90 & 23 \\ 36 & 155 & 46 \end{pmatrix} = \begin{pmatrix} 8 & 15 & 23 \\ 4 & 25 & 0 \end{pmatrix}$ HOWDY_.

- e.g. using the Cayley-Hamilton method, find A^8, $A = \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix}$

$A^8 = c_0 I + c_1 A$.

If λ is an eigenvalue of A, $c_0 + c_1\lambda = \lambda^8$.

A is lower triangular, $\lambda = -1, 2$

$$\begin{aligned} c_0 - c_1 &= 1 \\ c_0 + 2c_1 &= 256 \\ \hline 3c_1 &= 255. \\ c_1 = 85,\ & c_0 = 86. \end{aligned}$$

$A^8 = 86I + 85A$.

- last time - parity check code.
  - Hamming code.
  - experiment, outcome, sample space.
  - discrete / continuous.
  - event.
  - intersection $E \cap F$.
  - mutually exclusive.

Let E and F be events. Then their <u>union</u>, $E \cup F$,

is the event that E or F (or both) will occur.

It is the set of all outcomes lying in at least one

of E or F.

- e.g. $E = \{1, 3, 5\}$, $F = \{2, 4, 6\}$, $G = \{3, 6\}$.

  $E \cup F = \{1, 2, 3, 4, 5, 6\}$, $E \cup G = \{1, 3, 5, 6\}$, $F \cup G = \{2, 3, 4, 6\}$.

$E \cup E = E$, $E \cup S = S$, $E \cup \emptyset = E$.

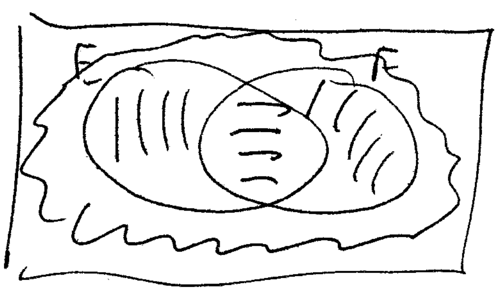
Let E be an event. Then its <u>complement</u>, $\bar{E}$ is the event that E does not occur. It is the set of all outcomes in S that are not in E.

- e.g. E, F, G as above. $\bar{E} = F$, $\bar{F} = E$, $\bar{G} = \{1,2,4,5\}$.

$\bar{S} = \phi$, $\bar{\phi} = S$, $\overline{(\bar{E})} = E$.

We can illustrate using a <u>Venn diagram</u>. The sample space is indicated with a rectangle. Events are circles inside it. The intersections are the overlapping regions.

- e.g.



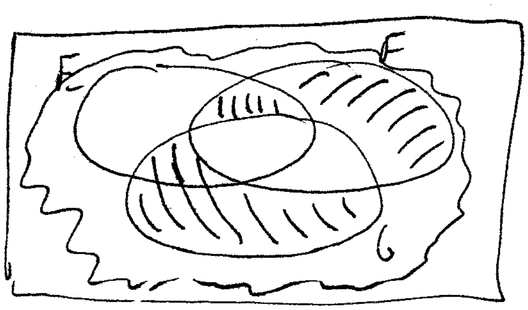
$\equiv$ = $E \cap F$

$|||$ = $E \cap \bar{F}$

$///$ = $\bar{E} \cap F$

= $\bar{E} \cap \bar{F} = \overline{E \cup F}$.

- e.g.



$||||$ = $E \cap F \cap \bar{G}$.

$///$ = $\bar{E} \cap F \cap \bar{G}$ ←

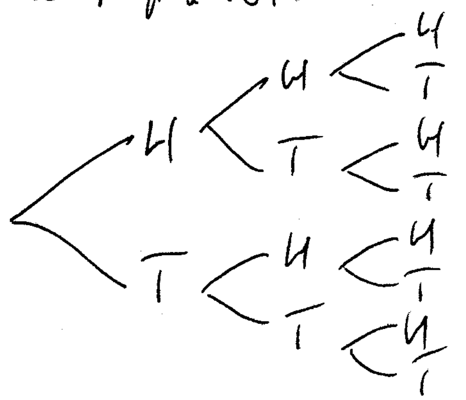
$\backslash\backslash\backslash$ = $G \cap \bar{F}$.

= $\bar{E} \cap \bar{F} \cap \bar{G} = \overline{E \cup F \cup G}$.

$E \cap F = F \cap E$.

A multistage experiment can be illustrated with a <u>tree diagram</u>.

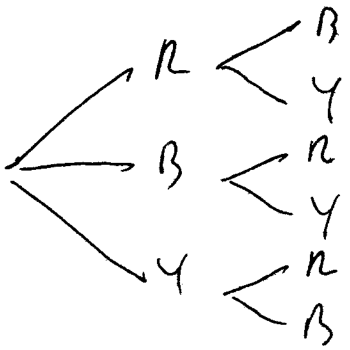
- e.g. we flip a coin 3 times.



HHH
HHT
HTH
HTT
THH
THT
TTH
TTT.

- e.g. a drawer contains a red sock, a blue sock, and a yellow sock.

  We draw 2 socks at random, without replacement.



RB
RY
BR
BY
YR
YB.

MULTIPLICATION RULE: Suppose an experiment has k stages.

Also, suppose there are n_i outcomes at stage i, no matter what happened before. Then the total number of possible outcomes is $n_1 n_2 \cdots n_k$.

- e.g. Flip a balanced coin 9 times. Total # outcomes: $2^9 = 512$.

- e.g. our Xena fan club has 10 members. We must select a president, VP, treasurer. How many ways? $10 \cdot 9 \cdot 8 = 720$.

- e.g. We have 6 Buffy action figures to line up on a shelf.

  # of ways? $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

Let n be a positive integer. Then

$n! := n(n-1)(n-2)\cdots(2)(1)$

"n factorial"

$0! = 1$.

- e.g. $5! = 5(4)(3)(2)(1) = 120$.

A <u>permutation</u> is an ordered list of elements drawn without replacement from a set.

Suppose we have a set of size n, and we want a permutation of length k. The number of possibilities is:

$P(n,k) = {}_nP_k = n(n-1)(n-2)\cdots(n-k+1)$.

$$= \frac{n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots(1)}{(n-k)(n-k-1)\cdots(1)}$$

$$= \frac{n!}{(n-k)!}$$

- e.g. $P(8,3) = 8\cdot 7\cdot 6 = \frac{8!}{5!} = 336$.

- e.g. our Xena fan club has 14 female, 12 male members

  We must select a pres, VP, treasurer.

  (i) How many ways?

  (ii) How many ways result in at least one man getting a job?

  (iii) How many ways result in a woman being president?

  (iv) How many ways result in Bob getting a job?

  (i) $P(26,3)$ (ii) $P(26,3) - P(14,3)$.

  (iii) $14\,P(25,2)$ (iv) $P(26,3) - P(25,3)$

- e.g. at a track meet, there are 15 male, 15 female competitors.

  We must award 1st through 5th place ribbons to each gender.

  (i) How many ways? (ii) How many ways result in Jane coming 4th?

  (i) $P(15,5)\,P(15,5)$ (ii) $P(15,5)\,1\,P(14,4)$

- e.g. a license plate has 4 letters followed by 3 digits

(i) # of possible plates? (ii) # of possible plates with no repetition?

(i) $26^4 10^3$ (ii) $P(26,4)\,P(10,3)$

03/02/18

- last time - union, complement.
  - Venn diagram.
  - tree diagram.
  - multiplication rule.
  - permutation. $P(n,k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$.

A <u>combination</u> is a subset of a particular size drawn from a set. Repetition is not allowed, and order does not matter.

$\#\text{perms} = (\#\text{combs})(\#\text{of orderings})$

$P(n,k) = (\#\text{combs})(k!)$

The number of combinations is $\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!k!}$

"n choose k"

- e.g. $\binom{6}{2} = \frac{6!}{4!2!} = \frac{720}{24 \cdot 2} = 15$.

$\binom{n}{1} = n$

$\binom{n}{n-1} = n.$

$\binom{n}{k} = \binom{n}{n-k}.$

$\binom{n}{n} = 1 = \binom{n}{0}.$

- e.g. in poker, we draw a 5-card hand randomly from the deck.

  (i) how many hands? (ii) how many hands contain 3 cards of one rank, but another ("full house").

  (i) $\binom{52}{5}$ (ii) $13\binom{4}{3}12\binom{4}{2}$

- e.g. we have 1000 batteries, 40 of them are dead. How many ways are there to pick 12 batteries and get 3 dead ones?

  $\binom{40}{3}\binom{960}{9}.$

- e.g. our Xena fan club has 30 members. We must select a pres, VP, treasurer, and an advisory committee of 5. How many ways to assign the positions?

  $P(30,3)\binom{27}{5} = \binom{30}{5}P(25,3)$

Let E be an event. Then its <u>probability</u>, $Pr(E)$, is a number with $0 \le Pr(E) \le 1$ indicating the likelihood of that event occurring. The higher, the more likely.

We write $N(E)$ for the number of outcomes in E.

If all outcomes for an experiment are equally likely, then for any event E, $Pr(E) = \dfrac{N(E)}{N(S)}$

- eg. we roll 2 balanced dice, find the prob. that we get a total of 10.

$S = \{11, 12, \ldots, 16, 21, 22, \ldots, 66\}$.

$E = \{46, 55, 64\}$.

$Pr(E) = \dfrac{N(E)}{N(S)} = \dfrac{3}{36} = \dfrac{1}{12}$.

- eg. In Lotto 6/49, we randomly select 6 numbers from 1 to 49. 6 winning numbers are randomly selected.

Find the prob of (i) matching all 6 numbers

(ii) matching 4 numbers

(c) $\dfrac{1}{\binom{49}{6}}$ (ii) $\dfrac{\binom{6}{4}\binom{43}{2}}{\binom{49}{6}}$

- e.g. In poker, find the probability of getting "two pairs".

QQ77K.

$$\frac{\binom{13}{2}\binom{4}{2}^2 44}{\binom{52}{5}}$$

- e.g. find the prob. of getting a "flush".

$$\frac{4\left(\binom{13}{5}-4\cdot 10\right)}{\binom{52}{5}}$$

rule out straight flush
- all one suit, all in sequence.

- e.g. find the prob. of getting exactly 2 kings, at least 2 queens, no clubs

$$\frac{\binom{3}{2}\left(1+\binom{3}{2}11(31)\right)}{\binom{52}{5}}.$$

- e.g. a license plate consists of 6 letters. Find the prob. that a randomly selected plate (i) has no repeated letters (ii) has at least one Q.

(i) $\dfrac{P(26,6)}{26^6}$ (ii) $\dfrac{26^6 - 25^6}{26^6}$

MARCH 5/18
APPLIED ANAL.

- last time
  - Combinations, $\binom{n}{k} = \frac{n!}{(n-k!)k!}$
  - Probability
  - equally likely outcomes $Pr(E) = \frac{N(E)}{N(S)}$

Axioms of Probability:

(i) $0 \le Pr(E) \le 1$, for all events E

(ii) $Pr(S) = 1$

(iii) if $E \cap F = \emptyset$, then $Pr(E \cup F) = Pr(E) + Pr(F)$

$E \cup \bar{E} = S$, $E \cap \bar{E} = \phi$

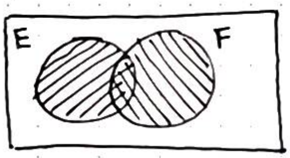
$Pr(E \cup \bar{E}) = Pr(E) + Pr(\bar{E})$

$Pr(S) = Pr(E) + Pr(\bar{E})$

$1 = Pr(E) + Pr(\bar{E})$

$Pr(\bar{E}) = (-Pr(E))$

Pr(E) = sum of the probabilities of the outcomes in E:

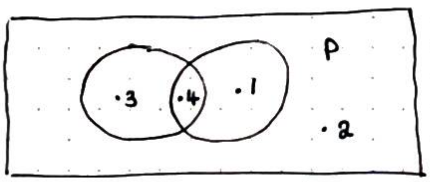


In general, $Pr(E \cup F) = Pr(E) + Pr(F) - Pr(E \cap F)$

- e.g. $Pr(E) = 5$, $Pr(F) = 6$, $Pr(E \cap F) = 8$, what is $Pr(E \cap F)$?
  $-8 = -5 + 6 - Pr(E \cap F)$, so $Pr(E \cap F) = 3$

We can put probability into the regions of Venn diagrams

- e.g. in a survey of 100 people, 70 liked cake, 50 liked pie, 40 liked both cake and pie. Find the probability that a randomly selected participant liked neither cake or pie.



.2

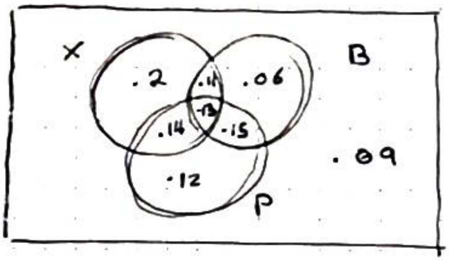
- e.g. in a survey of 100 people, 58 liked Xena, 46 liked Buffy, 54 liked South Park, 24 liked X + B, 27 liked X + P, 28 liked B + P, 13 liked all 3.

Find the probability that a randomly selected participant liked:

(i) Xena, nothing else $\rightarrow$ .2

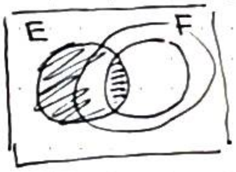
(ii) exactly one show $\rightarrow$ .38

(iii) Buffy + South Park $\longrightarrow$ .17 = 0.11 + 0.06



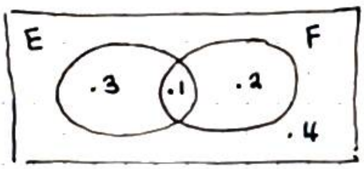
Let E and F be events, then the <u>conditional possibilities</u> of E given F is

$$Pr(E|F) = \frac{Pr(E \cap F)}{Pr(F)}$$



- e.g. $Pr(E) = .4$, $Pr(F) = .3$, $Pr(E \cap F) = .1$

Find (i) $Pr(E|F)$ (ii) $Pr(F|E)$ (iii) $Pr(\bar{E}|F)$



(i) $\dfrac{Pr(E \cap F)}{Pr(F)} = \dfrac{(.1)}{(.3)} = \dfrac{1}{3}$ (ii) $\dfrac{Pr(F \cap E)}{Pr(E)} = \dfrac{(0.1)}{(0.4)} = \dfrac{1}{4}$

(iii) $= \dfrac{Pr(\bar{E} \cap F)}{Pr(F)} = \dfrac{(.2)}{(.3)} = \dfrac{2}{3}$

Events E and F are <u>independent</u> if $Pr(E \cap F) = Pr(E)Pr(F)$

If $Pr(F) \neq 0$, this means $\dfrac{Pr(E \cap F)}{Pr(F)} = Pr(E)$

so, $Pr(E|F) = Pr(E)$

if $Pr(E) = 0$, $\dfrac{Pr(F \cap E)}{Pr(E)} = Pr(E)$

so $Pr(F|E) = Pr(E)$

- e.g. Flip a balanced coin twice

Let E be the event that we get heads on Flip 1

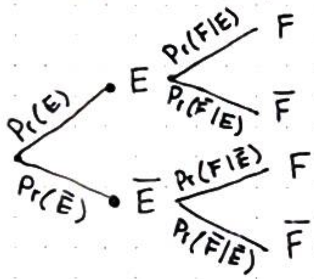
Let F " " Flip 2

$Pr(E) = Pr(F) = 1/2$

$\{HH, HT, TH, TT\}$ $Pr(E \cap F) = 1/4 = Pr(E)Pr(F)$

E and F are independent

Note that $Pr(E)Pr(F|E) = Pr(E \cap F)$



$Pr(E \cap F) = Pr(E)Pr(F|E)$

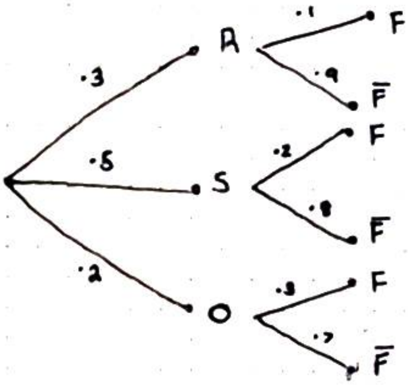
$Pr(E \cap \bar{F}) = Pr(E)Pr(\bar{F}|E)$

" "

" "

We can write conditional probability on the branches of a tree diagram, then multiply along the path to get the probability of the outcome.

- e.g. 30% of the cars in the parking lot are red 50% are silver, 20% are some other colour. Further suppose 10% of the red cars, 20% of the silver cars, 30% of the other cars have fuzzy dice.



$Pr(F) = (0.3)(0.1) + (0.5)(0.2) + \dots$
$\dots (0.2)(0.3) = 0.19$

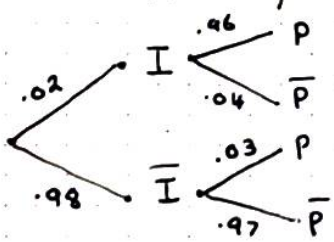
In Bayes Probability, we are given $Pr(E|F)$, we want to know $Pr(F|E)$. These can be solved with tree diagrams

- e.g. In the car problem, given that the car has fuzzy dice, find the prob. that it is red.

$$Pr(R|F) = \frac{Pr(R \cap F)}{Pr(F)} = \frac{(0.3)(0.1)}{(0.19)} = \frac{3}{19}$$

- e.g. 2% of the population has Xena fever. There is a test, but it has a 3% false positive, and a 4% false neg. Given that a randomly selected person tests positive, what is the probability that he has Xena fever?

I = infected, P = tests positive



$$Pr(I|P) = \frac{Pr(I \cap P)}{Pr(P)}$$

$$= \frac{(.02)(.96)}{(.02)(.96) + (.98)(.03)}$$

- e.g. Find the prob. of getting "one-pair" in poker

$$\rightarrow \frac{13\binom{4}{2}\binom{12}{3}4^3}{\binom{52}{5}}$$

- e.g. Find the prob. of getting at least 4 hearts, and exactly one 7

$$\rightarrow \frac{\binom{12}{4} + \binom{12}{3}36 + \binom{12}{4}3}{\binom{52}{5}}$$

Case 1: All hearts

- e.g. Find the probability that our hand will contain two kings, and exactly one club

$$\rightarrow \frac{3\binom{36}{3} + \binom{3}{2}12\binom{36}{2}}{\binom{52}{5}}$$

Case 1: Club is a king
Case 2: Club isn't a king

- e.g. Fizzbin: You get 7 randomly dealt cards
A royal Fizzbin is three of a kind and two pairs
Find the prob of getting one: KKKQQ44

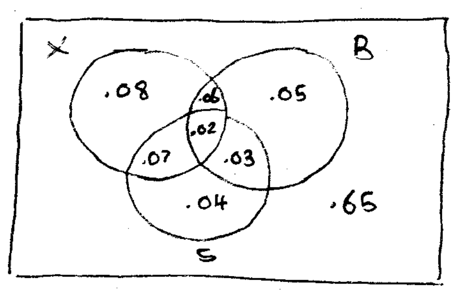
$$\rightarrow \frac{13\binom{4}{3}\binom{12}{2}\binom{4}{2}^2}{\binom{52}{7}}$$

- e.g. Our Xena Fan club has 23 members, including Bob. We must select a pres, tres. and vice pres. what are the odds of bob being president?

$$\frac{P(22,2)}{P(23,3)} = \frac{\cancel{22}\cdot\cancel{21}}{23\cdot\cancel{22}\cdot\cancel{21}} = \frac{1}{23}$$

- e.g. We ask 100 people about their favorite shows.
23 like Xena, 16 like Buffy, 16 like simpsons
8 like X + B, 9 like X + S, 5 like B + S
2 like all 3

(i) likes neither B or S



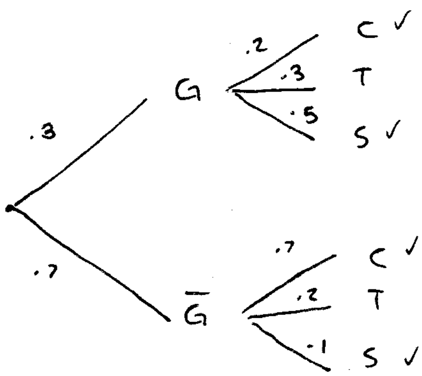
(i) $P_r(\bar{B} \cap \bar{S}) = 0.08 + 0.65$
$= 0.73$

(ii) $P_r(\bar{B} \mid X) = \dfrac{P_r(\bar{B} \cap X)}{P_r(X)}$

$\Rightarrow \dfrac{.08 \times .07}{.23} = \dfrac{15}{23}$

- 30% of the vehicles in the parking lot are green.
  - ↳ of the green vehicles, 20% cars, 30% trucks, 50% SUV
  - ↳ of non green vehicles, 70% cars, 20% trucks, 10% SUV

Given that a randomly selected vehicle is not a truck, Find the prob. that it is green



$$\Pr(G \mid \bar{T}) : \frac{(.3)(.2) + (.3)(.5)}{(.3)(.2) + (.3)(.5) + (.7)(.7) + \ldots + (.7)(.1)}$$

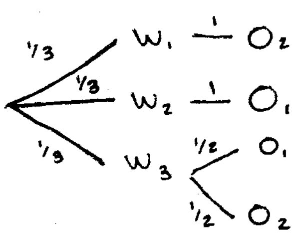
MARCH 10/18
APPLIED ANAL.

- Last time
  - axioms of probability
  - $Pr(\bar{E}) = 1 - Pr(E)$, $Pr(E \cup F) = Pr(E) + Pr(F) - Pr(E \cap F)$
  - $Pr(E) = \sum$ Prob of outcomes in E
  - Venn diagrams
  - Combined Prob. $Pr(E|F) = \dfrac{Pr(E \cap F)}{Pr(F)}$
  - independent
  - tree diagrams
  - Bayes' probabilities
- e.g. Monty-hall problem : You are shown three doors, behind one is a car, and the others goats. You choose door 3, Monty opens a door to reveal a goat. would you like to Keep door 3, or switch for another door? Switch!

W_i = door i is the winner

O_i = monty opens door i

We chose door 3.



$$Pr(W_3|O_1) = \frac{Pr(W_3 \cap O_1)}{Pr(O_1)} = \frac{(1/3)(1/2)}{(1/3)(1)+(1/3)(1/2)} = 1/3$$
$\hookrightarrow = 1/3$

$$Pr(W_2|O_1) = \frac{Pr(W_2 \cap O_1)}{Pr(O_1)} = \frac{(1/3)(1)}{(1/3)(1)+(1/3)(1/2)} = 2/3$$
$\hookrightarrow = 2/3$

Chapter 4 - Probability Distributions

A <u>random variable</u> X assigns a numerical value to each possible outcome of an experiment

- e.g. roll 2 balanced dice, let x = total
- e.g. Flip a coin 3 times, let x be the number of heads

A <u>probability distribution</u> assigns a probability $f(x)$ to each possible value x of X

$f(x) = Pr(X = x)$

We usually denote this with a table

- e.g. Flip a balanced coin 3 times, Let x = # of heads

x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

- e.g. roll 2 balanced dice, let x be the total

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	1/36	2/36	3/36	4/36	5/36	6/36	[illegible]/36	4/36	3/36	2/36	1/36

- e.g. Flip a balanced coin until head appears

x	1	2	3	...	n	...
$f(x)$	1/2	1/4	1/8	...	$\frac{1}{2^n}$	...

To get a valid distribution we must have:

$f(x) \geq 0$ for all x

$$\sum_{\text{all } x} f(x) = 1$$

- e.g.

x	-762	18	2394
$f(x)$	1/2	1/6	1/3

Valid

<u>Cumulative distribution</u> = $F(x) = Pr(X \leq x)$

$$\Rightarrow \sum_{y \leq x} f(y)$$

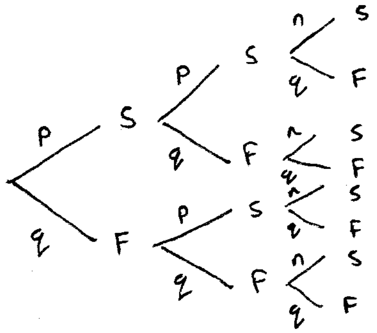
- e.g.

x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8
$F(x)$	1/8	1/2	3/8	1

A <u>Bernouli process</u> (or <u>Bernouli trials</u>) is a series of trials, with the following assumptions:

1) At each stage, there are two possibilities, success (S) and Failure (F)
2) The probability of success, P, is the sum of each stage
3) The stage is independent
4) The number of stages, n, is predetermined

We are interested in the number of success, x



# of success	Prob.
3	p^3
2	$p^2 \varepsilon$ ←
2	$p^2 \varepsilon$ ←
1	$p \varepsilon^2$
2	$p^2 \varepsilon$ ←
1	$p \varepsilon^2$
1	$p \varepsilon^2$
0	ε^3

In general each prob. leading to x-successes in n-attempts

has Probability $p^x q^{n-x} = p^x (1-p)^{n-x}$

There are $\binom{n}{x}$

Last time
- Monty-hall
- Discrete random variable
- discrete probability distribution
- $f(x) \geq 0$, $\sum_{all\ x} f(x) = 1$
- Cumulative distribution $F(x)$
- Bernouli trials
- Binomial distribution $\binom{n}{x} p^x (1-p)^{n-x}$

- e.g. you take a test with 13 derivative questions, 7 integral questions. Each question has 5 possible answers. You feel that you have a 90% chance on each derivative question, you must guess on the integral questions. You're offered \$500 if you score 95% or better. Find your probs. of winning.

$Pr\left(\frac{13}{13} \text{ deriv.}, \frac{6}{7} \text{ int}\right) + Pr\left(\frac{12}{13} \text{ deriv.}, \frac{7}{7} \text{ integral}\right) + Pr\left(\frac{13}{13} \text{ deriv.}, \frac{7}{7} \text{ int}\right)$

$= (.9)^{13}\binom{7}{6}(.2)^6(.8)^1 + \binom{13}{12}(.9)^{12}(.1)^1(.2)^7 + (.9)^{13}(.2)^7$

(*) The <u>cumulative binomial distribution</u> is

$B(x,n,p) = \sum_{i=0}^{x} b(i,n,p)$

There is a table - You may not get it in the exam

- e.g. You perform 10 Bernouli trials with $p = .4$. In terms of B, Find:

(i) The prob of getting at most 3 successes
(ii) " " fewer than 3 successes
(iii) " " more than 3 successes
(iv) " " at least 3 successes
(v) " " at least 3, no more than 5 successes
(vi) " " 3 successes

(i) $B(3, 10, .4)$ (ii) $B(2, 10, .4)$ (iii) $1-B(3, 10, .4)$
(iv) $1-B(2, 10, .4)$ (v) $B(5, 10, .4) - B(2, 10, .4)$
(vi) $B(3, 10, .4) - B(2, 10, .4)$

Hypergeometric distribution: We have N items and a are special. We randomly select n items without replacement. The prob. of getting x special items in our sample is:

$$L(x; n, a, N) = \frac{\binom{a}{x}\binom{N-a}{n-x}}{\binom{N}{n}}$$

- e.g. Find the prob of getting exactly two clubs in a particular hand

$$\frac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}}$$

- e.g. Suppose we have 1000 batteries 70 of which are dead. If we randomly select 40 batteries Find the prob. of getting 8 dead ones if:
  (i) We remove each battery after test
  (ii) We replace each dead battery

→ (i) $\frac{\binom{70}{8}\binom{930}{32}}{\binom{1000}{40}}$ (ii) $\binom{40}{8}\left(\frac{70}{1000}\right)^8\left(\frac{930}{1000}\right)^{32}$

As $n << N$, the hypergeometric can be approximated by a second | - general, the approximation is good at $n \leq \frac{N}{10}$

- e.g. We have a large collection of light bulbs, 30% of which are burnt out. We randomly select 100 bulbs, find the Prob. of getting 25 burnt out bulbs. We don't know N, must approx. using binomial

$\binom{100}{25}(.3)^{25}(.7)^{75}$

(Sections 2.5, 2.6)

Suppose we have numbers $x_1, x_2, \ldots, x_n$

The Sample mean is $\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n}$

- e.g. 1, 7, 3, 2 $\bar{x} = \frac{1+7+3+2}{4} = \frac{13}{4}$

If $x_1 \le x_2 \le x_3 \le \dots \le x_n$, then the sample median

$$\begin{cases} x_{(n+1)/2}, & n \text{ odd} \\ \dfrac{x_{n/2} + x_{(n+1)}}{2}, & n \text{ even} \end{cases}$$

- e.g. 2, 5, 3, 1, 8, 14, 17

  1 2 3 5 8 14 17 → 5 median

- e.g. 2 3 8 1 4 6

  1 2 3 4 6 8 → $\frac{3+4}{2} = 7/2$ median

- e.g. 60 60 60 60 60 60
  20 20 20 100 100 100 } 60 median

The <u>sample variance</u> is $s^2 = \dfrac{\sum_{i=1}^{n} (x_i - \bar{x}_i)^2}{n-1}$

- e.g. 1, 3, 4, 8 $\bar{x} = \dfrac{1+3+4+8}{4} = 4$

$$s^2 = \frac{(1-4)^2 + (3-4)^2 + (4-4)^2 + (8-4)^2}{3} = \frac{26}{3}$$

This measures how spread out the values are

The <u>sample standard deviation</u> is $s = \sqrt{s^2}$

- e.g. $s = \sqrt{26/3}$

Suppose we have $x_1 \le x_2 \le x_3 \dots \le x_n$

Let $0 < p < 1$. Then the $(100p)^{th}$ <u>percentile</u> is a number η so that

(a) at least $100p$ % of the data is $\le \eta_c$ and

(b) at least $100(1-p)$ % of the data is $\ge \eta$

To find the $(100p)^{-th}$ percentile, calculate np

If np is not an integer, round up to the next <u>larger</u> integer, use x_k

If np is an integer, use $\dfrac{x_{np} + x_{np+1}}{2}$

- e.g. 1, 2, 8, 11, 14, 19, 23, 27, 30, 42

Find (i) 23rd percentile (ii) 80th percentile

(i) $np = 10(.23) = 2.3$, $x_3 = 8$

(ii) $np = 10(.8) = 8$, $\frac{x_8 + x_9}{2} = \frac{27 + 30}{2} = \frac{57}{2}$

The quartiles are the 25th, 50th, 75th percentiles

Q_1 : 25th

Q_2 : 50th - median

Q_3 : 75th

- e.g. 1, 8, 2, 7, 3, 5

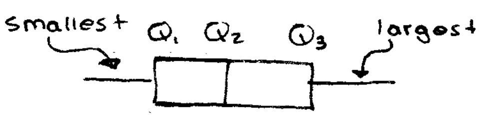
1, 2, 3, 5, 7, 8

Q_1 : $np = 6(.25) = 1.5$, $x_2 = 2$

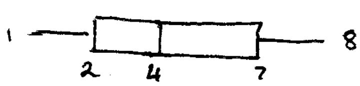
Q_2 : $\frac{3+5}{2} = 4$ (median)

Q_3 : $np = 6(.75) = 4.5$, $x_5 = 7$

A <u>boxplot</u> can be used to represent the data



$\Longrightarrow$



The <u>interquartile range</u> is $Q_3 - Q_1$

- e.g. $7 - 2 = 5$

MARCH 14TH / 18
Applied Analysis

- Last time
  - cumulative distribution
  - hypergeometric $\dfrac{\binom{a}{x}\binom{N-a}{n-x}}{\binom{N}{n}}$
  - Sample mean, median
  - sample variance, sample standard deviation
  - percentiles, quartiles
  - box plot, interquartile range

Let x be a discrete random variable, then is <u>mean</u> (or <u>expected value</u>) is $\mu = E(x) = \sum_{\text{all } x} x f(x)$

- e.g.

x	1	2	3
$f(x)$	$1/2$	$1/4$	$1/4$

$\mu = 1(1/2) + 2(1/4) + 3(1/4) = 7/4$

We say that a gain is Fair if our expected gain is $\emptyset$

- e.g. In a lottery 1000 tickets are sold. 1 First prize of \$500, and there are 10 prizes of \$100. How much should a ticket cost to make it fair?

Let c be a cost of a ticket, Let x be our

x	$500-c$	$100-c$	$-c$
$f(x)$	$\frac{1}{1000}$	$\frac{10}{1000}$	$\frac{989}{1000}$

$\emptyset = \mu = (500-c)\left(\frac{1}{1000}\right) + (100-c)\left(\frac{10}{1000}\right) - c\left(\frac{989}{1000}\right)$

$\emptyset = 1500 - 1000c, \quad c = 50$

- e.g. In chuck-a-luck, you pick a number between 1 → 6. Three balanced dice are rolled. If your number comes up at least once, you gain a dollar for each time it appears. If your number doesn't come up, you lose a dollar. Find expected gain/loss.

x	3	2	1	-1
$f(x)$	$(1/6)^3$	$\binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2$ $= \frac{15}{6^3}$	$\binom{3}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$ $= 75/6^3$	$(5/6)^3$ $= \frac{125}{216}$

$\mu = 3\left(\frac{1}{216}\right) + 2\left(\frac{15}{216}\right) + 1\left(\frac{75}{216}\right) - 1\left(\frac{125}{216}\right) = \frac{-17}{216}$

	1	2	3	4	5	6	
123 :	-1	-1	-1	+1	+1	+1	0
112 :	-2	-1	+1	+1	+1	+1	1
111 :	-3	+1	+1	+1	+1	+1	2

For a binomial : $\mu = np$

- e.g. Flip a balanced coin 80 times, expected # of heads : $80(\frac{1}{2}) = 40$

For hypergeometric : $\mu = n(\frac{a}{N})$

- e.g. We have 100 bottles of water, 20 of which are poisoned. If we randomly select 8 bottles, the expected number of poisoned ones is $8(\frac{20}{1000}) = 1.6$

We can measure how spread out the values of x tend to be

The <u>variance</u> is $\sigma^2 = \sum_{\text{all } x} (x-\mu)^2 f(x)$

- e.g.

x	4	8	12
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$\mu = 4(\frac{1}{4}) + 8(\frac{1}{2}) + 12(\frac{1}{4}) = 8$

$\sigma^2 = (4-8)^2(\frac{1}{4}) + (8-8)^2(\frac{1}{2}) + (12-8)^2(\frac{1}{4})$

$= 8$

The <u>standard deviation</u> is $\sigma = \sqrt{\sigma^2}$

- e.g. above, $\sigma = \sqrt{8}$

For binomial; $\sigma^2 = npq = np(1-p)$

For hypergeometric; $\sigma^2 = n(\frac{a}{N})(-\frac{a}{N})(\frac{N-n}{N-1})$

- e.g. We have 1000 cars, 200 are green. If we randomly select 80 cars, Find the standard dev. in the number of cars if : (i) we sample w/o replacement

(ii) we replace each after sampling

(i) $\sqrt{80\left(\frac{200}{1000}\right)\left(\frac{800}{1000}\right)\left(\frac{920}{999}\right)}$ (ii) $\sqrt{80(.2)(.8)}$

<u>Chebyshev's Theorem:</u> Let x be a random variable with mew μ, standard deviation σ. Then, for any $k > 0$,

$Pr[|x-\mu| \geq k\sigma] \leq 1/k^2$

Proof: $\sigma^2 = \sum_{\text{all } k} (x-\mu)^2 f(x)$

$\geq \sum_{|x-\mu| \geq \sigma k} (x-\mu)^2 f(x)$

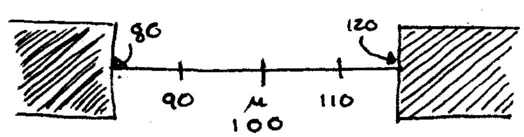
$\geq \sum_{|x-\mu| \geq \sigma k} k^2\sigma^2 f(x)$

$\therefore 1 \geq \sum_{|x-\mu| \geq k\sigma} k^2 f(x)$

$1/k^2 \geq \sum_{|x-\mu| \geq k\sigma} f(x)$

$Pr(|x-\mu| \geq k\sigma) \leq 1/k^2$

- e.g. X has mean 100 and standard deviation 10
What can we say about the prob that $x \leq 80$ or $x \geq 120$



$\mu = 100$
$\mu + k\sigma = 120$
$\sigma = 10$; $k = 2$

$Pr[(x-100(\geq 2(10)) \leq 1/2^2 = 1/4$

Prob is at most 1/4

$Pr(80 \leq x \leq 120) = 1 - Pr(x \leq 80 \text{ or } x \geq 120) \geq 1 - 1/4 = 3/4$

- last time: - $\mu = \sqrt{2}\,|x|$,
  - binomial, hypergeometric
  - variance σ^2, standard deviation σ
  - Chebyshev's thm: $\Pr(|x-\mu| \geq k\sigma) \leq \frac{1}{k^2}$
- e.g. we flip a balanced coin 10000 times, find out what Chebyshev says about the prob. of the proportion of heads being at most 45% or at least 55%

X = # of heads, $n = 10000$, $p = 1/2$

$\left|\frac{X}{n} - .5\right| \geq .05$

$|x - \underbrace{.5n}_{\mu}| \geq \underbrace{.05n}_{k\sigma}$

x is binomial, $\mu = np = .5n = 5000$

$k\sigma = .05n$, but $\sigma = \sqrt{np(1-p)} = \sqrt{n/4}$

$k\sigma = k(\sqrt{n/4}) = .05n$

$k = \frac{.05n}{\sqrt{n/4}} = \frac{.05\sqrt{n}}{1/2} = .1\sqrt{n} = .1(100) = 10$

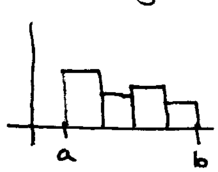
$\Pr(|x-\mu| \geq k\sigma) \leq 1/k^2$

prob $\leq 1/10^2 = .01$

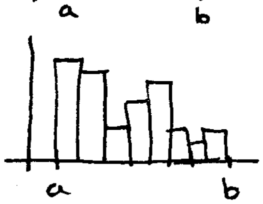
Chapter 5 - Probability Densities (5.1 - 5.6)

A continuous random variable X has values in $\mathbb{R}$ or some interval. We will not worry about the probability of x being a particular value. Instead, we look at the prob of x being in some interval.

Let us say that x takes on values in $[a, b]$



Area of each rectangle = prob. that x is in that interval



Refine
repeat

In the limits we assign a function $f(x)$ so that $Pr[c \le x \le d] = \int_c^d f(x)dx$

We call $f(x)$ the probability density function

Rules: (i) $f(x)$ is an I-grable function (?)

(ii) $f(x) \ge 0$ for all x

(iii) $\int_{-\infty}^{\infty} f(x)dx = 1$

If $f(x)$ is only defined on $[a,b]$, make it 0 everywhere else

- e.g. X has density function $f(x) = \begin{cases} kx^3, & x \in [0,1] \\ 0, & \text{elsewhere} \end{cases}$

(i) Find k (ii) Find $Pr(1/4 \le x \le 1/2)$ (iii) Find $Pr(x < 1/3)$

(iv) Find $Pr(x > 2/3)$ (v) Find $Pr(x = 1/2)$

→ (i) $1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 kx^3 dx = \frac{kx^4}{4}\Big|_0^1 = \frac{k}{4}$, so $k = 4$

(ii) $\int_{1/4}^{1/2} f(x)dx = \int_{1/4}^{1/2} 4x^3 dx = x^4\Big|_{1/4}^{1/2} = 1/16 - 1/256$

(iii) $\int_{-\infty}^{1/3} f(x)dx = \int_0^{1/3} 4x^3 dx = x^4\Big|_0^{1/3} = 1/81$

(iv) $\int_{2/3}^{\infty} f(x)dx = \int_{2/3}^{1} 4x^3 dx = x^4\Big|_{2/3}^{1} = 1 - 16/81$

(v) 0

- e.g. x has density function $f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & x \le 0 \end{cases}$

(i) Find $Pr(x < 4)$ (ii) Find $Pr(x > 2)$

→ (i) $\int_{-\infty}^{4} f(x)dx \Rightarrow \int_0^4 3e^{-3x} \Rightarrow -e^{-3x}\Big|_0^4 = -e^{-12} + 1$

(ii) $\int_2^{\infty} f(x)dx \Rightarrow \int_2^{\infty} 3e^{-3x} \Rightarrow -e^{-3x}\Big|_2^{\infty} = 0 - (-e^{-6}) = e^{-6}$

The mew of x is $\mu = \int_{-\infty}^{\infty} x f(x)dx$

The variance is $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

The standard deviation is $\sigma = \sqrt{\sigma^2}$

- e.g. $f(x) = \begin{cases} 4x^3, & x \in [0,1] \\ 0, & \text{elsewhere} \end{cases}$

$\mu = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 4x^3 dx = \frac{4x^5}{5}\Big|_0^1 = 4/5$

$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \int_0^1 (x - 4/5)^2 (4x^3) dx$

$\dots \Rightarrow 4/6 - \frac{32}{25} + \frac{64}{100}$

The distribution factor is $F(x) = Pr(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$

- e.g. In the above example,

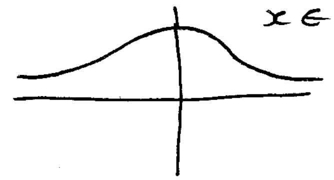
$F(x) : \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} 4t^3\,dt = t^4 \Big|_0^x = x^4, \; x \in [0,1]$

If $x < 0$, $F(x) = 0$. If $x > 1$, $F(x) = 1$

x has normal distribution with mew μ, standard deviation σ if it's density function is :

$x \in \mathbb{R}$

$$f(x) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right) e^{-(x-\mu)^2/(2\sigma^2)}$$



This function cant be integrated, but we can approximate as closely as needed.

Z is standard normal if it is normal with $\mu = 0$, $\sigma = 1$

We have the distribution function $F(z) = Pr(Z \leq z)$ in a table.

- e.g. Find (i) $Pr(Z < 1.36)$ (ii) $Pr(Z > -0.82)$
(iii) $Pr(0.34 \leq Z \leq 1.20)$

→ (i) $F(1.36) = 0.9131$

(ii) $1 - Pr(Z < -0.82) \Rightarrow 1 - F(-0.82)$

$\Rightarrow 1 - 0.2061 = 0.7939$

(iii) $Pr(Z < 1.20) - Pr(Z < 0.34)$

$\Rightarrow F(1.20) - [\,Pr(Z < 0.34)]$

$\Rightarrow F(1.20) - [\,F(0.34)]$

$\Rightarrow 0.8849 - 0.6331 = 0.2518$

- e.g. Find (i) $Pr(-1 \leq Z \leq 1)$ (ii) $Pr(-2 \leq Z \leq 2)$
(iii) $Pr(-3 \leq Z \leq 3)$

(i) $Pr(Z < 1) - Pr(Z < -1) = F(1) - F(-1) = 0.6826$

(ii) $Pr(Z < 2) - Pr(Z < -2) = F(2) - F(-2) = 0.9544$

(iii) $Pr(Z < 3) - Pr(Z < -3) = F(3) - F(-3) = 0.9974$

- e.g. Find a so that $Pr(Z > a) = .26$

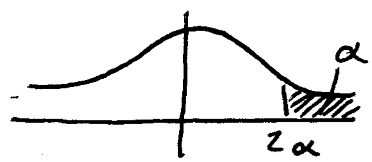
$Pr(Z \leq a) = .74 \Rightarrow F(.64) = .7389, \; F(.65) = .7454$

$a = .645$

MARCH 19/18
APPLIED ANAL

Last time - Continuous random variable
- Probability density Function $\int_c^d f(x)dx = Pr(x \in [c,d])$
- $f(x) \geq 0$, $\int_{-\infty}^{\infty} f(x) = 1$
- mean, variance, standard deviation
- distribution function
- normal distribution
- standard normal

Let $0 < \alpha < 1$, then z_α is a number such that $Pr(Z > z_\alpha) = \alpha$



$Pr(Z \leq z_\alpha) = 1 - \alpha$, To Find z_α we look inside the table for a number near $1-\alpha$

- e.g. Find (i) $z_{.01}$ (ii) $z_{.05}$ (iii) $z_{.83}$

(i) $F(2.32) = .9888$, $F(2.33) = .9901$, $z_{.01} = 2.325$

(ii) $F(1.64) = .9495$, $F(1.65) = .9505$, $z_{.05} = 1.645$

(iii) $F(-0.95) = .1711$, $F(-0.96) = .1685$, $z_{.83} = -0.955$

Suppose x is normal with new μ, standard deviation σ
Then we can standardize : $Z = \frac{x-\mu}{\sigma}$ is standard normal

$$Pr(a \leq X \leq b) = Pr\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right)$$
$$= F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$

- e.g Let x be normal with mean 70, standard dev. 5 :

Find (i) $Pr(x \leq 80)$ (ii) $Pr(x \leq 66)$ (iii) $Pr(69 \leq x \leq 72)$

(i) $F\left(\frac{80-70}{5}\right) = F(2) = .9972$

(ii) $1 - Pr(x \leq 66) = 1 - F\left(\frac{66-70}{2}\right) = 1 - F(-0.8) = 1 - 0.2119 = 0.7881$

(iii) $F\left(\frac{72-70}{2}\right) - F\left(\frac{69-70}{2}\right) = F(.4) - F(.2) = .6554 - .4201 = .2347$

- e.g. Let x be normal, $\mu = 100$, $\sigma = 10$, Find
(i) $P_r(x < 80)$ (ii) a so that $P_r(x > a) = .3$
(i) $F\left(\frac{80-100}{10}\right) = F(-2) = 0.0228$
(ii) $P_r(x \le a) = .7$, so $F\left(\frac{a-100}{10}\right) = 0.7$
$F(0.525) = 0.7$, so $\frac{a-100}{10} = 0.525$, so $a = 105.25$

- e.g. We have a machine that fills jars with jelly beans. Our machine can put in amounts that we normally distributed with $\sigma = 3g$
How should we set μ so that only 2% of the jars contain less than 500g of jelly beans?

$P_r(x < 500) = .02$
$P_r\left(z < \frac{500-\mu}{3}\right) = .02$
$F(-2.055) = .02$, so $\frac{500-\mu}{3} = -2.055$, $\mu = 506$

Let x be a binomial, then

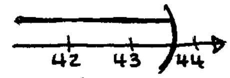
$$\frac{x - np}{\sqrt{np(1-p)}} \longrightarrow Z \text{ as } n \to \infty$$

If $np \ge 15$ and $n(1-p) \ge 15$, then it is reasonable to use a normal aprox. to a binomial
We must adjust our endpoints ± 0.5 to include or exclude the endpoint.

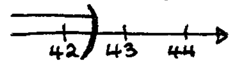
- e.g. Flip a balanced coin 100 times, count the heads.
Find the prob of getting:
(i) at most 43 heads
(ii) fewer than 43 heads
(iii) at least 43 heads
(iv) more than 43 heads
(v) at least 38, no more than 43 heads
(vi) exactly 43 heads (using normal approx.)

$$Z = \frac{x - np}{\sqrt{np(1-p)}} = \frac{x - 100(1/2)}{\sqrt{100(1/2)(1/2)}} = \frac{x-50}{5}$$

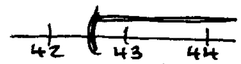
(i) $F\left(\frac{43.5-50}{5}\right) = F(-1.3) = .0968$



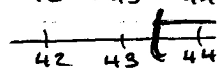
(ii) $F\left(\frac{42.5-50}{5}\right) = F(-1.5) = .0668$



(iii) $1 - F\left(\frac{42.5-50}{5}\right) = 1 - .0668 = .9332$

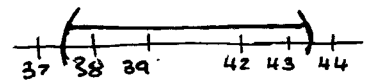


(iv) $1 - F\left(\frac{43.5-50}{5}\right) = 1 - .0968 = .9032$



(v) $F\left(\frac{43.5-50}{5}\right) - F\left(\frac{37.5-50}{5}\right)$

$\Rightarrow 0.0968 - F(-2.5) = .0968 - .0062 = .0906$



(vi) $F\left(\frac{43.5-50}{5}\right) - F\left(\frac{42.5-50}{5}\right)$

$\Rightarrow .0968 - .0668 = .03$

- e.g. Roll a balanced die 720 times, Find the prob. of getting exactly 110 fours

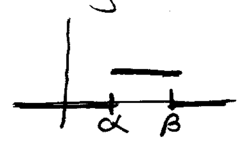
$$Z = \frac{x-np}{\sqrt{np(1-p)}} = \frac{x - 720(1/6)}{\sqrt{720(1/6)(5/6)}} = \frac{x-120}{10}$$

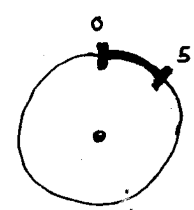
$$F\left(\frac{110.5-120}{10}\right) - F\left(\frac{109.5-120}{10}\right)$$

$\rightarrow F(-0.95) - F(-1.05) = .1711 - .16109 = .0242$

The <u>uniform distribution</u> has density function

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & x \in [\alpha, \beta] \\ 0, & \text{elsewhere} \end{cases}$$





- e.g. We take a wheel with radius 10 cm. We have painted a 5cm strip around the circumferance orange we spin the wheel radially on its axis. Blindfolded, I throw a dart and (...?) the circumference of the wheel. Find the prob. I hit the orange strip.

$$f(x) = \begin{cases} \frac{1}{10\pi}, & x \in (0, 10\pi) \\ 0, & \text{elsewhere} \end{cases}$$

$$Pr(0 \le x \le 5) = \int_0^5 \frac{1}{10\pi}\,dx = \frac{5}{10\pi}$$

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_\alpha^\beta \frac{x}{\beta-\alpha}\,dx = \frac{x^2}{2(\beta-\alpha)}\Big|_\alpha^\beta = \frac{\beta^2-\alpha^2}{2(\beta-\alpha)}$$

$$= \frac{\cancel{(\beta-\alpha)}(\beta+\alpha)}{(2)\cancel{(\beta-\alpha)}} = \frac{\beta+\alpha}{2}$$

$$\sigma^2 = \frac{(\beta-\alpha)^2}{12}$$

MARCH 21/18
APPLIED ANAL.

- last time
  - Z_α $P_r(Z > Z_\alpha) = \alpha$
  - standardizing $Z = \frac{x-\mu}{\sigma}$
  - normal approx. to
    $Z = \frac{x-np}{\sqrt{np(1-p)}}$, adjust endpoints $\pm 1/2$
  - uniform distribution

Let $\alpha \in \mathbb{R}$, $\beta > 0$, we say that x has log-normal dist.
if its density function is :

$$f(b) = \begin{cases} \frac{1}{\sqrt{2\pi}\,\beta}\left(x^{-1}e^{-((\ln x)-\alpha)^2/(2\beta^2)}\right), & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$\ln x$ is normal with mean α, standard deviation β

$$P_r(a \leq x \leq b) = F\left(\frac{\ln b-\alpha}{\beta}\right) - F\left(\frac{\ln a - \alpha}{\beta}\right)$$

- e.g. x has log-normal distribution, $\alpha = 5$, $\beta = 2$
  Find $P_r(e^4 \leq x \leq e^7)$

$$F\left(\frac{\ln(e^7)-5}{2}\right) - F\left(\frac{\ln(e^4)-5}{2}\right) = F\left(\frac{7-5}{2}\right) - F\left(\frac{4-5}{2}\right)$$

$$= F(1) - F(-0.5) = 0.8413 - 0.3085 = 0.5328$$

$$\mu = e^{\alpha+(\beta/2)}, \quad \sigma^2 = (e^{2\alpha+\beta^2})(e^{(\beta^*}-1)$$

Chapter 7

A population is a collection of numbers

The population is normal if upon selecting a number at random and letting it be x, the variable x is a normal variable.

The population mean μ is the mean of x

The population variance σ^2 is the variance of x

The population standard deviation is $\sigma = \sqrt{\sigma^2}$

A Parameter is a number determined from a population (e.g. μ, σ^2)

A statistic is a number calculated from a random sample. (e.g. $\bar{x}$, s^2)

A statistic is called an <u>unbiased estimator</u> if the expected value of the statistic equals the parameter

- e.g. $\bar{x}$ is an unbiased estimator to μ

We want the estimator to be as <u>efficient</u> as possible, that is, to have a σ as small as possible.

$\bar{x}$ is an efficient estimator for μ.

If we have a sample of size n, the expected value of $\bar{x}$ is μ.

The standard deviation of $\bar{x}$ is $\frac{\sigma}{\sqrt{n}}$

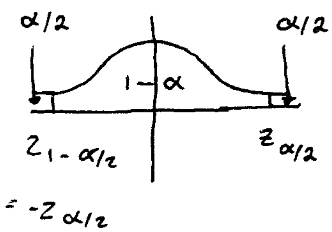
we concen ourselves with $\mu, \bar{x}$

$\bar{x}$ is our <u>point estimate</u> for μ

If x is normal, so is $\bar{x}$, and so

$Z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ ... is standard normal.

For any α, $0 < \alpha < 1$



$$Pr[-z_{\alpha/2} \le Z \le z_{\alpha/2}] = 1-\alpha$$

$$Pr[|Z| \le z_{\alpha/2}] = 1-\alpha$$

$$Pr\left[\left|\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right| < z_{\alpha/2}\right] = 1-\alpha$$

$$Pr\left[|\bar{x}-\mu| \le \frac{z_{\alpha/2}\sigma}{\sqrt{n}}\right] = 1-\alpha$$

Let $E = \frac{z_{\alpha/2}\sigma}{\sqrt{n}}$ we call this the <u>maximum error of the estimate</u>

(with probability $1-\alpha$)

- e.g. We have a normal population with standard dev. zero. We take a radom sample of size 100. Find the max error of the estimate for μ with probability .99

$1-\alpha = .99$, so $\alpha = .01$, $z_{.005} = 2.575$

$$E = \frac{z_{\alpha/2}\sigma}{\sqrt{n}} = \frac{2.575(20)}{\sqrt{100}} = 5.15$$

Suppose E is predetermined, and we want to find n

$E = \frac{z_{\alpha/2}\sigma}{\sqrt{n}}$, so $\sqrt{n} = \frac{z_{\alpha/2}\sigma}{E}$, and $n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2$

- e.g. Suppose we have a normal population with $\sigma = 10$

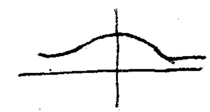
How large a sample do we need to get a max. error of the estimate of 4, with prob. 0.99.

$n = \left(\frac{2.575\,(10)}{4}\right)^2$, 41.4 round up to 42

If σ is unknown, we want to use s to approximate σ

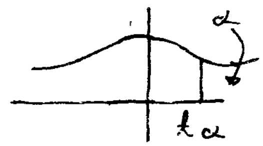
Let $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$ - This has t-distribution with

$\nu = n-1$ degrees of freedom.



t is symmetric, non-zero, Standard deviation is not 1

$t \rightarrow Z$ as $n \rightarrow \infty$



The table gives t_α values

$Pr(t > t_\alpha) = \alpha$

$t_{1-\alpha} = -t_\alpha$

We will use $E = \dfrac{t_{\alpha/2}\, s}{\sqrt{n}}$

- e.g. We have a normal population. we take a random sample of size 25. Find the max. error of the estimate with prob. .95, assuming that the standard dev. is 12.

$1 - \alpha = .95$, so $\alpha = .05$, $t_{.025} = 2.064$ $(\nu = 24)$

$E = \dfrac{2.064\,(12)}{\sqrt{25}}$

→ If $n \geq 30$ we use Z even if σ is unknown

→ If σ is known, $E = \dfrac{z_{\alpha/2}\,\sigma}{\sqrt{n}}$

→ If σ is unknown, $n < 30$, $E = \dfrac{t_{\alpha/2}\, s}{\sqrt{n}}$, $\nu = n-1$

→ If σ is unknown, $n \geq 30$, $E = \dfrac{z_{\alpha/2}\,\sigma}{\sqrt{n}}$

- e.g. We take a random sample of size 100 from a normal pop., we get a sample standard deviation of 7. Find the max error of the estimate with prob. 0.96

→ σ is unknown, $n \geq 30$, so $E = \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}}$

$1 - \alpha = 0.95$, $\alpha = 0.05$, $z_{.025} = 1.96$

$E = \frac{1.96\,(7)}{\sqrt{100}}$

T(1)

MARCH 23/18

APPLIED ANAL.

→ x has density function

$$f(x) = \begin{cases} K\sin x, & x \in [0, \pi] \\ 0, & \text{elsewhere} \end{cases}$$

(i) Find K

(ii) Find $Pr(x > \pi/4)$

↳ (i) $1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^{\pi} K\sin x = -K\cos x \Big|_0^{\pi} = -K(-1-1)$, $K = 1/2$

(ii) $\int_{\pi/4}^{\infty} f(x)dx = \int_{\pi/4}^{\pi} 1/2 \sin x\, dx = -1/2 \cos x \int_{\pi/4}^{\pi} = -1/2(-1 - 1/\sqrt{2})$

→ Find $Z_{.03}$

$Pr(Z > Z_{.03}) = .03$

$Pr(Z < Z_{.97}) = .97$

$F(1.88) = .9699$, $F(1.89) = .9706$

$Z_{.03} = 1.885$

→ x is normal, $\mu = 80$, $\sigma = 10$

Find (i) $Pr(x < 63)$

(ii) a so that $Pr(x < a) = 0.3$

↳ (i) $F\left(\frac{63-80}{10}\right) = F(-1.7) = 0.0446$ %

(ii) $F\left(\frac{a-80}{10}\right) = 0.3$

$F(-0.525) = 0.7$

$\frac{a-80}{10} = -0.525$, $a = 80 - 5.25$

→ x is normal, $\mu = 100$

$Pr(x > 120) = 0.1$

Find σ

$Pr(x < 120) = 0.9$

$F\left(\frac{120-100}{\sigma}\right) = 0.9 \rightarrow F(1.285) = 0.9$

$20/\sigma = 1.285 \Rightarrow \sigma = 20/1.285$

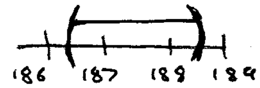
→ We Flip a balanced coin 400 times. Find the prob. of getting either 187 or 188 heads

(i) exactly

(ii) a normal approx.

↳ (i) $\binom{400}{187}(1/2)^{187}(1/2)^{213} + \binom{400}{188}(1/2)^{188}(1/2)^{212}$

(ii) $Z = \frac{x-np}{\sqrt{np(1-p)}} = \frac{x-400(1/2)}{\sqrt{400(1/2)(1/2)}} = \frac{x-200}{10}$



$F\left(\frac{188.5-200}{10}\right) - F\left(\frac{186.5-200}{10}\right) = F(-1.18) - F(-1.35)$

$= .1251 - .0885$

→ we take a random sample of size n From a normal population. What is the max. error of the estimate For μ if:

(i) $\sigma = 10$, $n = 16$

(ii) σ is unknown, $s = 12$, $n = 16$

(iii) σ is unknown, $s = 12$, $n = 1000$ → "nu"

↳ in each case with prob. .95

(i) as σ is known, $E = \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}} = \frac{z_{.025}(10)}{\sqrt{16}} = \frac{1.96(10)}{4}$

(ii) as σ is unknown, $n < 30$, $E = \frac{t_{\alpha/2}\,s}{\sqrt{n}} = \frac{t_{.025}(12)}{\sqrt{16}} = \frac{2.131(12)}{4}$

(iii) as σ is unknown, $n \geq 30$, $E = \frac{z_{\alpha/2}\,s}{\sqrt{n}} = \frac{z_{..}(12)}{\sqrt{1000}} = \frac{1.96(12)}{\sqrt{1000}}$

- e.g. when a normal pop $\sigma = 30$, how large a sample do we need to get a max. error in the estimate of μ to be 2 with Prob. 0.95?

$n = \left(\frac{z_{\alpha/2}\,\sigma}{E}\right)^2 = \left(\frac{1.96(30)}{2}\right)^2 = 864.36$ so 865

MARCH 23 /18
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Last time - log-normal distribution
- population
- parameter, Statistic
- max error of the estimate → σ known, $E = \frac{z_{\alpha/2}\sigma}{\sqrt{n}}$
→ σ unknown, $n < 30$, $E = \frac{t_{\alpha/2}\sigma}{\sqrt{n}}$; $\nu = n-1$
→ σ unknown, $n \geq 30$, $E = \frac{z_{\alpha/2} S}{\sqrt{n}}$

- if we have σ, and we want a particular n, $n = \left(\frac{z_{\alpha/2}\,\sigma}{\sqrt{n}}\right)^2$

we have seen that $Pr(|\bar{x} - \mu| \leq E) = 1 - \alpha$

The prob. that $\bar{x}$ is between $\mu - E$ and $\mu + E$ is $1 - \alpha$

we would like to say that the prob. that μ is between $\bar{x} - E$ and $\bar{x} + E$ is $1 - \alpha$, but μ is either there or it isn't!

Instead we will say that we are $100(1-\alpha)\%$ confident that μ is between $\bar{x} - E$ and $\bar{x} + E$

We call $(\bar{x} - E, \bar{x} + E)$ the $100(1-\alpha)\%$ confidence interval.

- e.g we take a random sample of size 12 from a normal pop. and get a sample mean of 40 and a sample standard deviation of 7. Find the 99% confidence interval for μ.

As σ is unknown, $n < 30$, $E = \frac{t_{\alpha/2}\, S}{\sqrt{n}} = \frac{t_{.005}(7)}{\sqrt{12}} = \frac{3.106(7)}{\sqrt{12}}$ $(\nu = 11)$

$\left[\bar{x} - \sqrt{2}, \bar{x} + \sqrt{2}\right] = \left[40 - \frac{3.106(7)}{\sqrt{2}}, 40 + \frac{3.106(7)}{\sqrt{2}}\right]$

- e.g. Same problem, except that we know $\sigma = 7$

As σ is known, $\sqrt{2} = \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}} = \frac{2.005(7)}{\sqrt{12}} = \frac{2.575(7)}{\sqrt{12}}$

$\left[\bar{x} - \sqrt{2}, \bar{x} + \sqrt{2}\right] = \left[40 - \frac{2.575(7)}{\sqrt{2}}, 40 + \frac{2.575(7)}{\sqrt{2}}\right]$

We want to make hypothesis concerning means

The null hypothesis, H_0, is the hypothesis that we are setting up to see if we can reject it.

The alternative hypothesis, H_1, is the logical negative of H_0

It is what we are trying to establish.

We will either reject H_0 or Fail to reject H_0

Never accept H_0!

A type I error occurs if we reject H_0, even though it is true

A type II error occurs if we had to reject H_0, even though it is False

	H_0 true	H_0 false
reject H_0	Type I error	✓
do not reject H_0	✓	Type II error

Possible hypothesis:

$H_0: \mu \leq \mu_0$ | $H_0: \mu \geq \mu_0$ | $H_0: \mu = \mu_0$

$H_1: \mu > \mu_0$ | $H_1: \mu < \mu_0$ | $H_1: \mu \neq \mu_0$

The possibility of a type I error (assuming $\mu = \mu_0$) is called the level of significanse ...

The probability of a type II error, for some ... is denoted β.

In most cases, α will be given and we design the experiment accordingly. But we can have:

- e.g. Suppose we have a normal population with $\sigma = 10$

We wish to test this: $H_0: \mu \geq 50$, $H_1: \mu \leq 50$

We have a sample of size 25 and reject H_0 if $\bar{x} < 47$. Find:

(i) Find α (ii) Find β if $\mu = 45$

↳ (i) Assuming $\mu = 50$, we reject (incorrectly) is $\bar{x} < 47$

$$Pr[\bar{x} < 47] = Pr\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} < \frac{47-50}{10/\sqrt{25}}\right] = Pr(Z < -1.5) = 0.0668$$

(ii) $$Pr(\bar{x} > 47) = 1 - Pr(\bar{x} < 47) = 1 - Pr\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \leq \frac{47-45}{10/\sqrt{25}}\right)$$

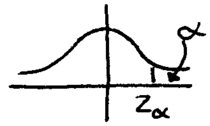
$$\Rightarrow 1 - Pr(Z \leq 1) = 1 - 0.8413 = 0.1587$$

Suppose we are given α, and for now, that σ is known

$H_0 = \mu \le \mu_0$ | Let $Z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$; we will reject that if

$H_1 = \mu > \mu_0$

we reject if $Z > Z_\alpha$



- e.g. the number of marshmallows in boxes of Krusty O's is normally distributed with $\sigma = 10$. Krusty claims that the number of marshmallows per box is on average at most 60. We randomly select 100 boxes and put a sample mean of 63. Test the claim at a .05 level of significance

$H_0 : \mu \le 60$ As σ is known, let $Z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ and

$H_1 : \mu > 60$ reject if $Z > Z_{.05}$

$$Z = \frac{63 - 60}{10/\sqrt{100}} = 3 \; ; \; Z_{.05} = 1.645$$

As $Z > Z_{.05}$ we reject the claim of a .05 level of significance.

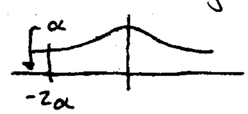
The book likes to do things like: $H_0 : \mu = 50 \leftarrow H_0 : \mu \le 50$

$H_1 : \mu \ge 50 \leftarrow$ correct

MARCH 26/18
APPLIED ANAL.

- last time
  - confidence interval for μ $[\bar{x} - E, \bar{x} + E]$
  - hypothesis testing - null/alternative hypothesis
  - Type I/II errors
  - level of significance α
  - usually, we get α, set up experiment
  - σ is known, $Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$
    $H_0 : \mu \le \mu_0$
    $H_1 : \mu > \mu_0$

$H_0 : \mu \ge \mu_1$ $\quad Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ and reject the null hypothesis
$H_1 : \mu < \mu_0$ $\quad$ if Z is "too negative"



reject that is $Z < -Z_\alpha$

- e.g. the number of jagged metal Krusty-O's in boxes of cereal are normally distributed with a standard deviation of 5. Krusty claims that the average number per box is at least 40. We randomly select 100 boxes and get a sample mean of 38.

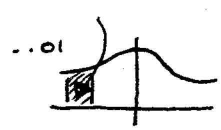
Test the claim at a .01 level of significance

$H_0 : \mu \ge 40$ / as σ is known, $Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ and we
$H_1 : \mu < 40$ / reject the null hypothesis $Z < Z_{.01}$

$$Z = \frac{38 - 40}{5/\sqrt{100}} = -4 \; ; \; Z_{.01} = 2.325$$

As $Z < -Z_{.01}$, we reject the claim at a .01 level of significance

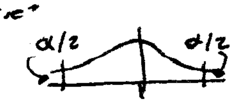
The last hypothesis of tests are called <u>one-handed tests</u>



The <u>two-handed test</u> is:

$H_0 : \mu = \mu_0$ / $Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ and we reject that if
$H_1 : \mu \ne \mu_0$ /

Z is "too positive" or "too negative"



we reject if $Z > Z_{\alpha/2}$

or $Z < -Z_{\alpha/2}$

That is, reject if $|Z| > Z_{\alpha/2}$

- e.g. The height of Canadian women are normally distributed with $\sigma = 5$cm. Statistics Canada claims that the mean height is 166cm. We randomly select 25 women and get a sample mean of 164 cm

Test the claim at a .01 level of significance

$H_0 : \mu = 166$ / As σ is known, $Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ and reject

$H_1 : \mu \neq 166$ / that if $|Z| > Z_{.005}$

$Z = \frac{164 - 166}{5/\sqrt{25}} = -2$, $Z_{.005} = 2.575$

As $|Z| \not> Z_{.005}$, we cannot reject the claim at a .01 Level of Significance

If σ is unknown, we use S to approximate it

If σ is unknown, $n \geq 30$, $Z = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}$, and proceed as before

If σ is unknown, $n < 30$, $t = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}$, and test it

$t > t_\alpha$, $t < -t_\alpha$, $|t| > t_{\alpha/2}$

- e.g. Krusty claims that the average box of Krusty O's contains at least 200 mg of vitamins. We randomly select n boxes. we get a sample mean of 197 mg and a sample standard deviation of 10 mg. Take the claim at a .05 level of sig: (i) $n = 25$ (ii) $n = 100$

$H_0 = \mu \geq 200$ (i) as σ is unknown, $n < 30$

$H_1 = \mu < 200$ Let $t = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}$ and reject H_0 if $t < -t_{.05}$

$t = \frac{197 - 200}{10/\sqrt{25}} = -1.5$, As $\nu = 24$, $t_{.05} = 1.711$

As $t \not< -t_{.05}$, we cannot reject the claim at a .05 level of significance

(ii) As σ is unknown, $n \geq 30$, $Z = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}$ and reject that if $Z < Z_{.05}$

$Z = \frac{197 - 200}{10/\sqrt{100}} = -3$, $Z_{.05} = 1.645$

As $Z < -Z_{.05}$, we reject the claim at a .05 level of significance

Suppose we are performing a two-handed test.
Also assume σ is known, (but this doesn't matter)
We let $Z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$, and reject if $|Z| > Z_{\alpha/2}$

$$\left|\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right| > Z_{\alpha/2}$$

$$|\bar{x}-\mu_0| > \frac{Z_{\alpha/2}\,\sigma}{\sqrt{n}} = E$$

We reject that if $\mu_0 \notin [\bar{x}-E, \bar{x}+E]$
That is, we reject H_0 if μ_0 is not in the $100(1-\alpha)\%$ confidence interval. <u>Only for two-handed tests</u>!

-e.g. Statistics Canada claims that the average height for Canadian men is 177 cm. We randomly select 25 men and get a sample mean at 178 and a sample standard deviation of 5. Test the claim at a .01 level of significance.

$H_0 = \mu = 177$ As σ is unknown, $\sigma < 30$

$H_1 = \mu \neq 177$ $E = \frac{t_{\alpha/2}\, s}{\sqrt{n}} = \frac{t_{.005}(5)}{\sqrt{25}} = 2.797$ $(2 = 24)$

The 99% confidence interval for μ is :

$[\bar{x}-E, \bar{x}+E] = [178-2.297, 178+2.297] = [175.203, 180.797]$

As 177 lies in the confidence interval, we cannot reject the claim at a .01 level of significance.

MARCH 28 /18
APPLIED ANAL.

- Last time – hypothesis tests for μ
  - if σ is known, $Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$

$H_0: \mu \le \mu_0$ reject that if $Z > Z_\alpha$
$H_1: \mu > \mu_0$

$H_0: \mu \ge \mu_0$ reject that if $Z < -Z_\alpha$
$H_1: \mu < \mu_0$

$H_0: \mu = \mu_0$ reject that if $|Z| > Z_{\alpha/2}$
$H_1: \mu \ne \mu_0$

- what if σ is unknown?
- two handed test: Can use Confidence interval!

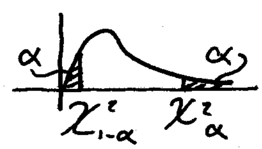
Chapter 9 – Inferences Concerning Variances (9.1 – 9.3)

Normal populations we want to know about σ^2 (or σ)
We take a random sample of size n
Normally, we use S^2 to estimate σ^2 (or s to estimate σ).
If n is small, we can use the sample range, $\mathbb{R}$ = biggest – smallest
Then $\sigma \simeq \mathbb{R}/d_2$

- e.g. we take a random sample: 8, 23, 7, 14, 11, 12, 9
$n = 23 - 7 = 16$, $d_2 = 2.704$, $\sigma \simeq 16/2.704$

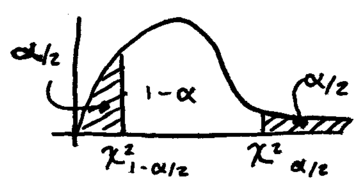
Let us discuss S^2
Let $\chi^2 = \frac{(n-1)S^2}{\sigma^2}$. This has chi-space distribution
with $\nu = n - 1$

A table gives χ^2_α values when
$P_r(\chi^2 > \chi^2_\alpha) = \alpha$



with probability $1-\alpha$:

$$\chi^2_{1-\alpha/2} \le \chi^2 \le \chi^2_{\alpha/2}$$
$$\chi^2_{1-\alpha/2} \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{\alpha/2}$$
$$\frac{1}{\chi^2_{\alpha/2}} \le \frac{\sigma^2}{(n-1)S^2} \le \frac{1}{\chi^2_{\alpha/2}}$$
$$\frac{(n-1)S^2}{\chi^2_{\alpha/2}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{\alpha/2}}$$



The $100(1-\alpha)\%$ confidence interval for σ^2 is :

$$\left[\frac{(n-1)s^2}{\chi^2_{\alpha/2}} , \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}} \right]$$

For σ :

$$\left[\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}}} , \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}} \right]$$

- e.g. We take a random sample of 21 boxes of Krusty O's and Find that the max number of jagged metal Krusty O's per box is 30 with a sample standard deviation of 8. Find a 95% confidence interval for the standard deviation in all boxes.

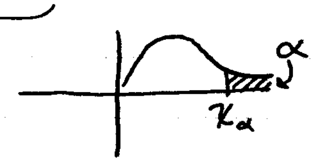
$$\left[\sqrt{\frac{20(8)^2}{\chi^2_{.025}}} , \sqrt{\frac{20(8)^2}{\chi^2_{.975}}} \right] = \left[\sqrt{\frac{20(8)^2}{34.170}} , \sqrt{\frac{20(8)^2}{9.591}} \right]$$

Hypothesis testing, for σ^2 (or σ):

$H_0 : \sigma^2 \le \sigma_0^2$ $\quad \chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$, and reject that if χ^2 is too large

$H_1 : \sigma^2 > \sigma_0^2$

if $\chi^2 > \chi^2_\alpha$ then reject.



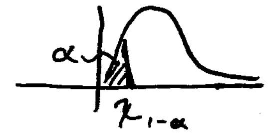
- e.g. A Pharmaceutical claims that the standard deviation in the amount of active ingredient in their pills is at most 5 micrograms. We randomly select 51 pills and get a sample standard deviation of 7 micrograms. Test the claim at a .05 level of significance.

$H_0 : \sigma^2 \le 5^2$ $\quad \chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$ and reject if $\chi^2 > \chi^2_{.05}$

$H_1 : \sigma^2 > 5^2$ $\quad \chi^2 = \frac{50(7^2)}{5^2} = 98$; $A_3 \approx 50$, $\chi^2_{.05} = 67.505$

$H_0 : \sigma^2 \geq \sigma_0^2$ $\qquad \chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$; reject if χ^2 is "too-large"

$H_1 : \sigma^2 < \sigma_0^2$ $\qquad$ reject if $\chi^2 < \chi^2_{1-\alpha}$



- e.g. A botanist needs leaves of various diameter. He insists upon a standard deviation of at least 10 cm. We randomly select 21 leaves and get a sample standard deviation of 9 cm. Test the claim that the leaves are satisfactory at a .01 level of significance

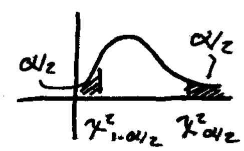
$H_0 : \sigma^2 \geq 10^2$ $\qquad \chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$, reject that if $\chi^2 < \chi^2_{.99}$

$H_1 : \sigma^2 < 10^2$ $\qquad \chi^2 = \dfrac{20(9)^2}{10^2} = 16.2$. As $\nu = 20$, $\chi^2_{.99} = 8.26$

As $\chi^2 \nless \chi^2_{.99}$ we cannot reject the claim at a .01 level of significance

$H_0 : \sigma^2 = \sigma_0^2$ $\qquad \chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$; reject that if χ^2 is "too big" or "too small"

$H_1 : \sigma^2 \neq \sigma_0^2$



Reject that if $\chi^2 > \chi^2_{\alpha/2}$ or $\chi^2 < \chi^2_{1-\alpha/2}$

-e.g. Stats Canada claims that the standard deviation in the length of Canadian men is 5cm. We randomly select 101 men and get a sample standard deviation of 4 cm. Test the claim at a .05 level of significance.

$H_0 : \sigma^2 = \sigma^2$ $\qquad \chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$; reject that if χ^2 is

$H_1 : \sigma^2 \neq \sigma^2$ $\qquad$ either $> \chi^2$ ($\chi^2 > \chi^2_{.025}$ or $\chi^2 < \chi^2_{.975}$)

As $\nu = 100$, $\chi^2_{.025} = 129.561$

$\chi^2_{.975} = 79.222$

As $\chi^2 < \chi^2_{.975}$, we reject the claim at a .05 level of significance.

Last time :
- Sample size ≤ 10, can use sample range
- usually, use s^2 to estimate σ^2
- chi-square distribution
- confidence interval for σ^2: $\left[\frac{(n-1)s^2}{\chi^2_{\alpha/2}}, \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}\right]$
- hypothesis testing $\chi^2 = \frac{(n-1)s^2}{\sigma_0}$

If we have a two tailed test, we can use a confidence interval to perform the test.

- e.g. Statistics Canada claims that the standard deviation in the heights of Canadian women is 5cm. We randomly select 11 women, and put a sample standard deviation at 6cm. Test the claim at a .05 level of significance.

$H_0 = \sigma^2 = 5^2$
$H_1 = \sigma^2 \neq 5^2$

The $100(1-\alpha)\% = 95\%$ confidence interval for σ^2 is $\left[\frac{10(6)^2}{\chi^2_{.025}}, \frac{10(6)^2}{\chi^2_{.975}}\right]$

$$\Rightarrow \left[\frac{10(6)^2}{20.483}, \frac{10(6)^2}{3.247}\right] = (17.58, 110.87)$$

As 5^2 lies in the 95% confidence interval, we cannot reject the claim at a .05 level of confidence.

- Suppose we have two normal populations, and we want to compare their variances, $Pop_1 = \sigma_1^2$
$Pop_2 = \sigma_2^2$

If we make the assumption that the variances are the ... and we take a random sample of size n_1 from Pop_1, n_2 from pop_2, and let $F = \frac{S_1^2}{S_2^2}$

Then F has distribution with $\nu_1 = n_1 - 1$, $\nu_2 = n_2 - 1$

We have $F_{.01}$ and $F_{.05}$ values.

$Pr(F > F_\alpha) = \alpha$

$$\left.\begin{array}{l} H_0: \sigma_1^2 \leq \sigma_2^2 \\ H_1: \sigma_1^2 > \sigma_2^2 \end{array}\right\} F = \frac{S_1^2}{S_2^2}, \text{ reject that if } F > F_\alpha$$

$$\left.\begin{array}{l} H_0: \sigma_1^2 \geq \sigma_2^2 \\ H_1: \sigma_1^2 < \sigma_2^2 \end{array}\right\} \text{ swap pops } 1 + 2$$

$H_0 : \sigma_1^2 = \sigma_2^2$ } If necessary, swap so that $S_1^2 \geq S_2^2$
$H_1 : \sigma_1^2 \neq \sigma_2^2$ } Repeat that if $F > F_{\alpha/2}$

- e.g. Krusty claims that the standard deviation in the number of marshmallows in boxes of Lucky charms is at least as great as that in Krusty-O's. We randomly select 10 boxes of Lucky Charms and 8 boxes of Krusty O's

For the lucky charms, we get a sample standard deviation of 10, for the Krusty O's, 15.

Test the claim at a .05 level of significance.

Pop_1 = Krusty-O's Pop_2 = Lucky Charms

$H_0 : \sigma_1^2 \leq \sigma_2^2$ $F = \frac{S_1^2}{S_2^2}$ and reject that if $F > F_{.05}$

$H_1 : \sigma_1^2 > \sigma_2^2$

$$F = \frac{15^2}{10^2} = 2.25$$

As $\nu_1 = 7$, $\nu_2 = 9$, $F_{.05} = 3.29$

As $F \not> F_{.05}$, we cannot reject the claim at a .05 level of significance.

Chapter 11 – Repression Analysis

Let x and y be random variables

y will depend upon x plus a randomness factor

$y = f(x) + \varepsilon$, when f is a function and ε is a random variable with non-0. We call $f(x)$ the <u>regression curve</u>. Finding it is called <u>repression analysis</u>.

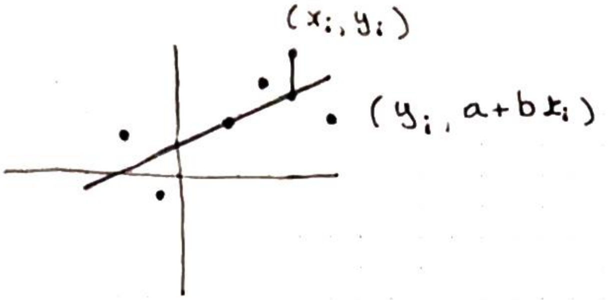
We can get an idea by taking a random sample. $(x_1, y_1), \ldots, (x_n, y_n)$, and drawing a scatterplot.

In <u>linear regression</u>, we have $y = \alpha + \beta x + \varepsilon$, $\alpha, \beta \in \mathbb{R}$

We will find the <u>line of best fit</u> for our data,

$\tilde{y} = a + bx$

We will use the <u>method of least squares</u>



We want to minimize the sum of the squares of the vertical distance from the points to the line

$$\sum_{i=1}^{n} (y_i - (a+bx_i))^2$$

$$a : \sum_{i=1}^{n} 2(y_i - (a+bx_i))(-1) = 0$$

$$\sum_{i=1}^{n} a + \sum_{i=1}^{n} bx_i = \sum_{i=1}^{n} y_i$$

$$na + \left(\sum_{i=1}^{n} x_i\right) b = \sum_{i=1}^{n} y_i \quad (1)$$

$$b : \sum_{i=1}^{n} 2(y_i - (a+bx_i))(-x_i) = 0$$

$$\sum_{i=1}^{n} ax_i + \sum_{i=1}^{n} bx_i^2 = \sum_{i=1}^{n} x_i y_i$$

$$\left(\sum_{i=1}^{n} x_i\right) a + \left(\sum_{i=1}^{n} x_i^2\right) b = \sum_{i=1}^{n} x_i y_i \quad (2)$$

Solve (1), (2) for a, b

- e.g. Find the line of best fit for:

x_i	1	2	3
y_i	7	3	1

$\rightarrow na + \sum x_i b = \sum y_i$

(1) $3a + 6b = 11$

$\rightarrow \sum x_i a + \sum x_i^2 b = \sum x_i y_i$

(2) $6a + 14b = 16$

(2) $-$ 2(1) $\Rightarrow 2b = -6$, $b = -3$

Subbing into (1): $3a + 6(-3) = 11$, $a = \frac{29}{3}$

$$\tilde{y} = \frac{29}{3} - 3x$$

- in the above example, estimating when $x = 2.5$

$$\tilde{y} = \frac{29}{3} - 3(2.5)$$

- Find the line of best fit for:

x_i	1	2	3	4
y_i	2	4	5	7

→ $na + \sum x_i b = \sum y_i$

① $4a + 10b = 18$

→ $\sum x_i a + \sum x_i^2 b = \sum x_i y_i$

② $10a + 30b = 53$

② − 3① : $-2a = -1$

$a = 1/2$

Subbing into ① : $4(1/2) + 10b = 18$

$b = 1.6$

$\tilde{y} = (0.5) + (1.6)x$

April 6/18
Applied Anal.
TUTORIAL

- e.g. we have a normal population with $\sigma = 10$
Suppose we wish to test.

$H_0 : \mu = 100$
$H_1 : \mu > 100$

we will take a random sample of size 25 and reject H_0 if $\bar{x} > 104$
Find α.

We assume $\mu = 100$

$$Pr(\bar{x} > 104) = 1 - Pr(\bar{x} \leq 104)$$
$$= 1 - F\left(\frac{104-100}{10/\sqrt{25}}\right) = 1 - F(2) = 1 - 0.9772$$
$$= 0.0228$$

- e.g. binomial : bernouli trials (Flipping coin, rolling die) } replacement!

- e.g. At least 4 hearts, and 1 seven?

4 hearts, including 7
$\binom{12}{3} 36$

4 hearts, not including 7
$\binom{12}{4} 3$

5 hearts
$\binom{12}{4}$

$$\Rightarrow \frac{\binom{12}{3}36 + \binom{12}{4}3 + \binom{12}{4}}{\binom{52}{5}}$$

- e.g. Two kings and one club?

King of clubs, another king
$3\binom{36}{4}$

no king of clubs
$\binom{3}{2}(12)\binom{36}{2}$

$$\Rightarrow \frac{3\binom{36}{4} + \binom{3}{2}12\binom{36}{2}}{\binom{52}{5}}$$

- e.g. A bag contains 40 red marbles and 60 blue marbles. We reach into the bag and pull out 10 marbles. Find the prob that we get 3 red marbles.

$$\frac{\binom{80}{3}\binom{60}{7}}{\binom{100}{10}}$$

$N = 100$
$n = 10$
$G = 40$
$x = 3$

w/ replacement: $\binom{10}{3}\left(\frac{80}{100}\right)^3\left(\frac{60}{100}\right)^3$

- e.g. Roll a balanced die 20 times, count the threes
  (i) Find the prob. of at most 4 threes
  (ii) Find the prob. of at least 4 threes

  (i) $B(4; 20, 1/6)$

  (ii) $1 - B(3, -20, 1/6)$

- e.g. Krusty claims that the average box of Krusty-O's contains at least 60 marshmallows. We randomly select 10 boxes and put a sample mean of 56 and a sample standard deviation at 10.
  Test the claim at a .05 level of significance

$H_0 : \mu \geq 60$ As σ is unknown, $n < 30$

$H_1 : \mu < 60$ $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$, reject H_0 if $t < t_{.05}$

$t = \dfrac{56-60}{10/\sqrt{16}} = -1.6$, As $\nu = 15$ $t_{.05} = 1.753$

As $t \nless -t_{.05}$ we cannot reject the claim.

- e.g. Krusty claims that the standard deviation in the number of jagged metal Krusty O's per box is 5. We randomly select 41 boxes and get a standard deviation of 3.
  Test the claim at a .05 level of significance

$H_0 : \sigma^2 = 5^2$ $\chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$, reject if $\chi^2 > \chi^2_{.025}$

$H_1 : \sigma^2 \neq 5^2$

$\chi^2 = \dfrac{40(3)^2}{5^2} = 14.4$

As $\nu = 4$, $\chi^2_{.025} = 59.342$

$\chi^2_{.975} = 24.433$

As $\chi^2 = \chi^2_{.975}$, we reject H_0

- e.g. Find the line of best fit:

x_i	1	2	3	4
y_i	6	5	3	1

$na + \sum x_i b = \sum y_i$

$4a + 10b = 15$ (1)

$\sum x_i a + \sum x_i^2 b = \sum x_i y_i$

$10a + (1+4+9+16)b = (6+10+9+4)$

$10a + 30b = 29$ (2)

solving (1) with (2)

$(2) - 3(1) = -2a = -16$

$a = 8$

then, $b = -1.4$

$\tilde{y} = 8 - 1.4x$

April 6/18
Applied Anal.

- Last time - Confidence intervals for two-handed test
  - F-test for two variables
  - regression analysis $y = f(x) + \varepsilon$
  - $y = \alpha + \beta x + \varepsilon$
  - line of best fit: $\hat{y} = a + bx$

$$na + \sum x_i b = \sum y_i$$
$$\sum x_i a + \sum x_i^2 b = \sum x_i y_i$$

If our data is $(x_1, y_1), \dots, (x_n, y_n)$

$S_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})^2$, $S_{yy} = \sum_{i=1}^{n} (y_i - \bar{y})^2$, $S_{xy} = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$

$b = S_{xy}/S_{xx} \Rightarrow a = \bar{y} - b\bar{x}$

- e.g.

x_i	1	2	3
y_i	8	3	1

$\bar{x} = \frac{1+2+3}{3} = 2$, $S_{xx} = ((1-2)^2 + (2-2)^2 + (3-2)^2) = 2$

$\bar{y} = \frac{8+3+1}{3} = 4$, $S_{yy} = ((8-4)^2 + (3-4)^2 + (1-4)^2) = 26$

$S_{xy} = (1-2)(8-4) + (2-2)(3-4) + (3-2)(1-4) = -4 + 0 - 3 = -7$

$b = S_{xy}/S_{xx} \Rightarrow (-7)/(2) = -7/2$; $a = 4 - (\frac{-7}{2})(2) \Rightarrow 4 + 7 = 11$

$\hat{y} = 11 - \frac{7}{2}x$

Exponential regression: $y = \alpha\beta^x + \varepsilon$

Our best fit curve will be $\hat{y} = ab^x$

$\log\hat{y} = \log a + x\log b$

Let $c = \log a$, $d = \log b$, we take $\log\hat{y} = c + dx$

We perform linear regression with $(x_i, \log y_i)$

x_i	1	2	3
y_i	1	10	1000
$\log y_i$	0	1	3

$nc + \sum x_i d = \sum \log y_i$

$3c + 6d = 4$ ①

$\sum x_i c + \sum x_i^2 d = \sum x_i \log y_i$

$6c + 14d = 11$ ②

② - 2①: $2d = 3$, $d = 3/2$

then $c = -5/3$

$a = 10^c = 10^{-5/3}$, $b = 10^d = 10^{3/2}$

$\hat{y} = 10^{-5/3}(10^{3/2})^x$

<u>Power</u> repression : $y = \alpha x^{\beta} + \varepsilon$

our best Fit curve : $\hat{y} = ax^{b}$

$\log \hat{y} = \log c + b \log x$. Let $c = \log$

$\log \hat{y} = c + d \log x$ Perform linear regression with $(\log x_i, \log y_i)$

x_i	10	100	1000
y_i	1000	100	1
$\log x_i$	1	2	3
$\log y_i$	3	2	0

$nc + \sum \log x_i d = \sum \log y_i$

$3c + 6d = 5$ ①

$\sum \log x_i c + \sum \log x_i^2 d = \sum (\log x_i)(\log y_i)$

$6c + 14d = 7$ ②

② - 2① : $2d = -3$; $d = -3/2$, $3c + 6(\frac{-3}{2}) = 5$, $c = 14/3$

$a = 10^{c} \Rightarrow 10^{14/3}$; $b = d = -3/2$

$\hat{y} = 10^{14/3} x^{-3/2}$

<u>Reciprocal</u> regression : $y = \frac{1}{\alpha + \beta x} + \varepsilon$

Our best Fit curve is $\hat{y} = \frac{1}{a + bx}$

$1/\hat{y} = a + bx$. Perform linear regression with $(x_i, 1/y_i)$

- e.g.

x_i	1	2	3	4
y_i	1	1/3	1/4	1/5
$1/y_i$	1	3	4	5

$na + \sum x_i b = \sum 1/y_i$

$4a + 10b = 13$ ①

$\sum x_i a + \sum x_i^2 b = \sum \frac{x_i}{y_i}$

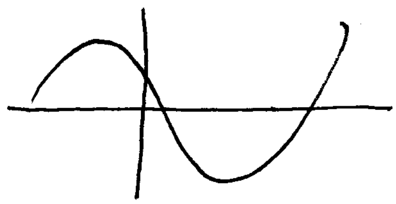
$10a + 30b = 39$ ②

3① - ② : $2a = 0$, $a = 0$; $b = 39/30 = 13/10$

$\hat{y} = \frac{1}{(13/10)x}$

Polynomial regression: $y = \beta_0 + \beta_1 x + \beta_2 x^2, \dots, \beta_p x^p + \varepsilon$

Our best fit curve is $\hat{y} = S_0 + S_1 x + S_2 x^2 + \dots + S_p x^p$



we will minimize the sum of the sequence of the vertical distance from the points to the curve.

$$\sum_{i=1}^{n} \left(y_i - (b_0 + b_1 x_i + b_2 x_i^2 + \dots + b_p x_i^p)\right)^2$$

Diff art b_0 : $\sum_{i=1}^{n} 2\left(y_i - (b_0 + b_1 x_i + b_2 x_i^2 + \dots + b_p x_i^p)\right)(-1) = 0$

$$n b_0 + \left(\sum_{i=1}^{n} x_i\right) b_1 + \left(\sum_{i=1}^{n} x_i^2\right) b_2 + \dots + \left(\sum_{i=1}^{n} x_i^p\right) b_p = \sum_{i=1}^{n} y_i \quad (1)$$

Diff art b_1 : $\sum_{i=1}^{n} 2\left(y_i - (b_0 + b_1 x_i + b_2 x_i^2 + \dots + b_p x_i^p)\right)(-x_i) = 0$

$$\left(\sum_{i=1}^{n} x_i\right) b_0 + \left(\sum_{i=1}^{n} x_i^2\right) b_1 + \left(\sum_{i=1}^{n} x_i^3\right) b_2 + \dots + \left(\sum_{i=1}^{n} x_i^{p+1}\right) b_p = \sum_{i=1}^{n} x_i y_i \quad (2)$$

$$\left(\sum_{i=1}^{n} x_i^2\right) b_0 + \left(\sum_{i=1}^{n} x_i^3\right) b_1 + \left(\sum_{i=1}^{n} x_i^4\right) b_2 + \dots + \left(\sum_{i=1}^{n} x_i^{p+2}\right) b_p = \sum_{i=1}^{n} x_i^2 y_i \quad (3)$$

$$\vdots$$

$$\left(\sum_{i=1}^{n} x_i^p\right) b_0 + \left(\sum_{i=1}^{n} x_i^{n+1}\right) b_1 + \left(\sum_{i=1}^{n} x_i^{n+2}\right) b_2 + \dots + \left(\sum_{i=1}^{n} x_i^{2p}\right) b_p = \sum_{i=1}^{n} x_i^p y_i \quad (p+1)$$

Solve for $b_0, b_1, \dots, b_p$

-e.g. Find the quadratic of best fit for :

x_i	1	2	3	4
y_i	-2	0	3	10

$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \varepsilon$

$\hat{y} = b_0 + b_1 x + b_2 x^2$

$n b_0 + \sum x_i b_1 + \sum x_i^2 b_2 = \sum y_i$

$4 b_0 + 10 b_1 + 30 b_2 = 11 \quad (1)$

$\sum x_i b_0 + \sum x_i^2 b_1 + \sum x_i^3 b_2 = \sum x_i y_i$

$10 b_0 + 30 b_1 + 100 b_2 = 47 \quad (2)$

$\sum x_i^2 b_0 + \sum x_i^3 b_1 + \sum x_i^4 b_2 = \sum x_i^2 y_i$

$30 b_0 + 100 b_1 + 354 b_2 = 185 \quad (3)$

- last time - S_{xx}, S_{yy}, S_{xy}, $b = \frac{S_{xy}}{S_{xx}}$, $a = \bar{y} - b\bar{x}$, $\hat{y} = a + bx$
  - exponential : $y = \alpha \beta^{y} + \varepsilon$
  - power : $y = \alpha x^{\beta} + \varepsilon$
  - reciprocal : $y = \frac{1}{\alpha + \beta x} + \varepsilon$
  - Polynomial repression : $y = \beta_0 + \beta_1 x + \beta_2 x^2 + \dots + \beta_p x^p + \varepsilon$

APRIL 9TH
APPLIED FINAL

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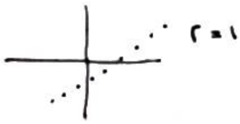
we have two random variables :, x, y

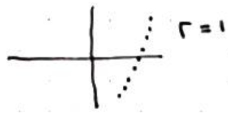
we take a random sample $(x_1, y_1) \dots (x_n, y_n)$

The sample correlation coefficient r, is a number, $-1 \le r \le 1$

Describing the direction and strength of the linear relationship between the variables.

$r > 1$: the points are all on a line with a positive slope.



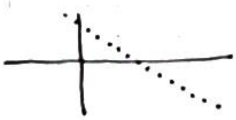


If r is near 1, there is a strong positive linear relationship

As x increases, y tends to increase

Strong, positive linear relationship

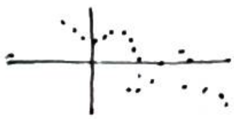
If $r = -1$, the points are all on a line with negative slope



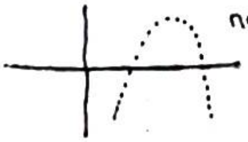
$r = -1$

If r is near -1, there is a strong negative linear relationship

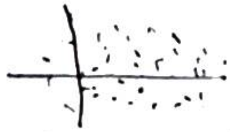
As x increases, y tends to decrease



If r is near $\varnothing$, there is a nonlinear or rectilinear or no relationship between the variables.



nonlinear relationship



$\bar{x}$ = mean of x_i, $\bar{y}$ = mean of y_i

S_x = standard deviation of x_i

S_y = standard deviation of y_i

$$r = \frac{1}{n-1} \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{S_x} \right) \left(\frac{y_i - \bar{y}}{S_y} \right)$$

- e.g.

x_i	1	2	3
y_i	1	3	8

$\bar{x} = \frac{1+2+3}{3} = 2$; $\bar{y} = \frac{1+3+8}{3} = 4$

$S_x^2 = \frac{(1-2)^2 + (2-2)^2 + (3-2)^2}{3-1} = 1$; $S_x = 1$

$S_y^2 = \frac{(1-4)^2 + (3-4)^2 + (8-4)^2}{3-1} = 13$, $S_y = \sqrt{13}$

$r = \left(\frac{1}{3-1}\right)\left(\frac{1}{1\sqrt{13}}\right)\left((1-2)(1-4) + (2-2)(3-4) + (3-2)(8-4)\right) = \frac{1}{2\sqrt{13}}(3+0+4) = \frac{7}{2\sqrt{13}}$

$r = \left(\frac{1}{n-1}\right)\left(\frac{1}{S_x S_y}\right)\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$

$S_x = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{S_{xx}}{n-1}}$

$S_y = \sqrt{\frac{\Sigma(y_i - \bar{y})^2}{n-1}} = \sqrt{\frac{S_{yy}}{n-1}}$

$r = \left(\frac{1}{n-1}\right)\left(\sqrt{\frac{n-1}{S_{xx}}}\right)\left(\sqrt{\frac{n-1}{S_{yy}}}\right)(S_{xy}) \Rightarrow \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}$

- e.g.

x_i	1	2	3
y_i	8	6	1

$\bar{x} = \frac{1+2+3}{3} = 2$; $\bar{y} = \frac{8+6+1}{3} = 5$

$S_{xx} = (1-2)^2 + (2-2)^2 + (3-2)^2 = 2$

$S_{yy} = (8-5)^2 + (6-5)^2 + (1-5)^2 = 26$

$S_{xy} = (1-2)(8-5) + (2-2)(6-5) + (3-2)(1-5) = -7$

$r = \frac{(-7)}{\sqrt{2 \cdot 26}} \cong -0.97$

Let ρ be the population correlation coefficient.

we will test $H_0 : \rho \le 0$, $H_1 : \rho > 0$; $H_0\ \rho \ge 0$, $H_1\ \rho < 0$; $H_0 : \rho = 0$, $H_1 : \rho \ne 0$

Let $\gamma = \frac{1}{2}\ln\left(\frac{1+r}{1-r}\right)$ Then $Z = \sqrt{n-3}\,\gamma$ is standard normal

Perform Z tests as normal ($Z > Z_\alpha$, $Z < -Z_\alpha$, $|Z| > Z_{\alpha/2}$)

- e.g. Suppose we have a sample of size 10 and got a sample correlation coefficient of $r = .732$. Test the hypothesis at a .01 level of significance.

$H_0 : \rho = 0$

$H_1 : \rho \ne 0$

$\gamma = \frac{1}{2}\ln\left(\frac{1+.732}{1-.732}\right) = .933$

$Z = \sqrt{10-3}\,\gamma = 2.47$

We will reject the null hypothesis if $|Z| > Z_{.005}$

But $Z_{.005} = 2.575$

As $|Z| \not> Z_{.005}$, we cannot reject the null hypothesis at a .01 level of significance.