$y'' - \frac{3}{x} y' + \frac{3}{x^2} y = 2x^2 e^x$

OCT. 30/17

Applied Anal.

Example Solve $x^2 y'' - 3xy' + 3y = 2x^4 e^x$

Solution: (1) Solve $x^2 y'' - 3xy' + 3y = \varnothing$

$y = x^m$ Auxiliary Eqn. $m^2 + (-3-1)m + 3 = \varnothing$

$m^2 - 4m + 3 = \varnothing$

$(m-1)(m-3) = \varnothing$ $m_1 = 1$; $m_2 = 3$

$y = c_1 x + c_2 x^3$; $y_1 = x$; $y_2 = x^3$

(2) Find a particular solution y_p

Variation of Parameters !

$y_p = u_1 y_1 + u_2 y_2$; $u_1' = \frac{W_1}{W}$; $u_2' = \frac{W_2}{W}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \Rightarrow \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^3 - x^3 = 2x^3$$

$$W_1 = \begin{vmatrix} \varnothing & y_2 \\ f(x) & y_2' \end{vmatrix} \Rightarrow \begin{vmatrix} \varnothing & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5 e^x$$

$$W_2 = \begin{vmatrix} y_1 & \varnothing \\ y_1' & f(x) \end{vmatrix} \Rightarrow \begin{vmatrix} x & \varnothing \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3 e^x$$

$$u_1' = \frac{W_1}{W} \Rightarrow \frac{-2x^5 e^x}{2\cancel{x^3}} = -x^2 e^x$$

$\int u_1' = -\int x^2 e^x \, dx \rightarrow$ (integration by parts, twice) $\longrightarrow -x^2e^x + 2xe^x - 2e^x$

$$u_2' = \frac{W_2}{W} \Rightarrow \frac{2\cancel{x^3}e^x}{\cancel{x^3}} = 2e^x$$

$\int u_2' = 2\int e^x \, dx \rightarrow 2e^x$

$y_p = 2x^2 e^x - 2xe^x$

(3) The general solution

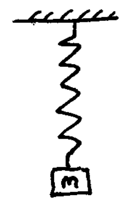
$y = c_1 x + c_2 x^3 + 2x^2 e^x - 2xe^x$

Nov. 1/17

Applied Anal.

3.8.1 Free undamped motion

$$m\frac{d^2x}{dt^2} = -kx$$



$$m\frac{d^2x}{dt^2} + kx = 0$$

- simple harmonic motion

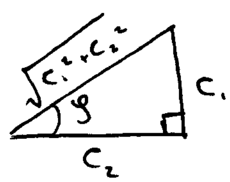
$$x'' + \frac{k}{m}x = 0 \quad ; \quad x'' + \omega^2 x = 0$$

$$x(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$$

Graph

$$x(t) = \sqrt{c_1^2 + c_2^2}\left(\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos(\omega t) + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin(\omega t)\right)$$

$$= \sqrt{c_1^2 + c_2^2}$$



$$\sin\varphi = \frac{c_1}{\sqrt{c_1^2+c_2^2}} \qquad \cos\varphi = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$$

$$\Rightarrow \sqrt{c_1^2 + c_2^2}\,(\sin\varphi\cos(\omega t) + \cos\varphi\sin(\omega t))$$

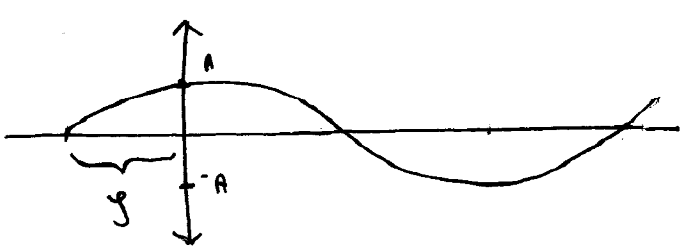
$$x(t) = \sqrt{c_1^2+c_2^2}\,\sin(\varphi + \omega t)$$

$A = \sqrt{c_1^2 + c_2^2}$ – amplitude

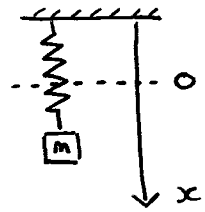
φ – Phase angle

$$x(t) = A\sin(\varphi + \omega t)$$

A - amplitude $\qquad$ φ - Phase angle



3.8.1 Free damped motion



damping force $= \beta \frac{dx}{dt}$ (the case)

Newton's Second Law:

$$m \frac{d^2x}{dt^2} = -kx - \beta \frac{dx}{dt}$$

$$m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{\beta}{m} \frac{dx}{dt} + \frac{k}{m} x = 0$$

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0$$

$x = e^{Mt}$ $\quad$ $\lambda = \beta / 2m$ $\quad$ $\omega = \sqrt{\frac{k}{m}}$

Auxiliary Equation:

$$M^2 + 2\lambda M + \omega^2 = 0$$

$$M = \frac{-2\lambda \pm \sqrt{(2\lambda)^2 - 4\omega^2}}{2} \Rightarrow -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

(I) $\lambda^2 - \omega^2 > 0$, (β is large when compared with k)

Overdamped

$$x(t) = C_1 e^{(-\lambda + \sqrt{\lambda^2 - \omega^2})t} + C_2 e^{(-\lambda - \sqrt{\lambda^2 - \omega^2})t}$$

$$x(t) = e^{-\lambda t}\left(C_1 e^{\sqrt{\lambda^2 - \omega^2}} + C_2 e^{-\sqrt{\lambda^2 - \omega^2}}\right)$$

(II) $\lambda = \omega$, $M_1 = M_2 = -\lambda$

Critically damped

$$x(t) = C_1 e^{-\lambda t} + C_2 e^{-\lambda t}$$

(III) $\lambda^2 - \omega^2 < 0$, underdamped (β is small)

$$x(t) = e^{-\lambda t}\left[\left(C_1 \cos(\sqrt{\lambda^2 - \omega^2}\, t\right) + \left(C_2 \sin(\sqrt{\lambda^2 - \omega^2}\, t\right)\right]$$

Example (overdamped motion)

A mass weighing 16 pounds stretches a spring 4 feet. Assuming a damping force numerically equal to 3 times the instantaneous velocity acts on the system. Determine the equation of motion if the mass is initially released from the equilibrium position with an upward velocity of 3 ft/s

Solution $m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$

$x(0) = 0 \quad ; \quad x'(0) = -3$

Hookes constant k : $F = ks$; $16 = k \cdot 4$; $k = 4$

$\beta = 3$ (damping force $= 3\frac{dx}{dt}$)

$m = \frac{W}{g} = \frac{16}{32} = \frac{1}{2}$

$$\boxed{\frac{1}{2}x'' + 3x' + 4x = 0 \quad ; \quad x(0) = 0, \quad x'(0) = -3}$$

$x^2 + 6x + 8 = 0 \quad ; \quad x = e^{Mt}$

Auxiliary Equation : $M^2 + 6M + 8 = 0$

$(M+2)(M+4) = 0 \quad ; \quad M_1 = -2, \ M_2 = -4$

$x(t) = C_1e^{-2t} + C_2e^{-4t}$

$x'(t) = -2C_1e^{-2t} - 4C_2e^{-4t}$

$x(0) = 0 \quad \Rightarrow \quad C_1 + C_2 = 0 \quad \Rightarrow \quad C_1 = -C_2$

$x'(0) = -3 \quad\quad -2C_1 - 4C_2 = -3 \quad\quad C_2 = 3/2 \quad\quad C_1 = -3/2$

$x(t) = -\frac{3}{2}e^{-2t} + \frac{3}{2}e^{-4t}$ is the solution

Example .... $\beta = 2$ ....

$\frac{1}{2}x'' + 2x' + 4x = 0$; $x(0) = 0$; $x'(0) = -3$

$x'' + 4x' + 8x = 0$ (Auxiliary equation $x = e^{Mt}$)

$M^2 + 4M + 8 = 0$

$$M = \frac{-4 \pm \sqrt{4^2 - (4)(1)(8)}}{2} = \frac{-4 \pm \sqrt{-16}}{2} = -2 \pm 2i$$

$x(t) = C_1 e^{-2t}\cos(2t) + C_2 e^{-2t}\sin(2t)$

$\left.\begin{array}{l} x(0) = 0 \\ x'(0) = -3 \end{array}\right\}$ $C_1 + C_2(0) = 0 \Rightarrow C_1 = 0$

$x(t) = C_2 e^{-2t}\sin(2t)$

$x'(t) = -2C_2 e^{-2t}\sin(2t) + C_2 e^{-2t}\,2\cos 2t$

$x'(0) = -3 \Rightarrow 0 + 2C_2 = -3$ $C_2 = -3/2$

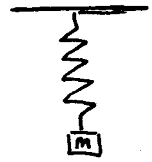
$x(t) = -3/2\, e^{-2t}\sin(2t)$ is the solution

3.8.3 Driven Motion

Nov. 3/17

Applied Anal.

External force $f(t)$



Newton's Second Law:

$$m\frac{d^2x}{dt^2} = \underbrace{-kx}_{\text{restoring F}} - \underbrace{\beta\frac{dx}{dt}}_{\text{damping F}} + \underbrace{f(t)}_{\text{external}}$$

$$\boxed{m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t)}$$

Example: (From overhead)

Solution: $m\frac{d^2x}{dt^2} + \underbrace{\beta\frac{dx}{dt}}_{\text{(damping)}} + \underbrace{kx}_{\text{(restoring)}} = \underbrace{f(t)}_{\text{(external)}}$

k: Hooke's Law, $F = ks$, $4 = k4$, $k = 1$

β: $\beta = \frac{1}{2}$

$m = \frac{W}{g} = \frac{4}{32} = \frac{1}{8}$

$f(t) = \frac{65}{56}\sin(t)$

$$\frac{1}{8}\frac{d^2x}{dt^2} + \frac{1}{2}\,dx/dt + x = 65/56\,\sin(t),$$

IVC: $x(0) = 3/7$, $x'(0) = 0$

Solve the IVP:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 8x = \frac{65}{7}\sin(t), \quad x(0) = 3/7, \quad x'(0) = 0$$

(I) Solve the associated homo eq'n:

$x'' + 4x' + 8x = 0$ $\qquad (x = e^{Mt})$

$M^2 + 4M + 8 = 0$

$$M = \frac{-4 \pm \sqrt{4^2 - 4(1)(8)}}{2(1)} = \frac{-4 \pm \sqrt{-16}}{2} \Rightarrow -2 \pm 2i$$

$\alpha = -2$

$\beta = 2$

$y = e^{-2t}(C_1\cos 2t + C_2\sin 2t)$

(II) Find a particular solution x_p :

$$x_p = A\cos t + B\sin t$$

$$x_p' = -A\sin t + B\cos t$$

$$x_p'' = -A\cos t - B\sin t$$

$$x_p'' + 4x_p' + 8x_p = \frac{65}{7}\sin t$$

$$(-A\cos t - B\sin t) + 4(-A\sin t + B\cos t) + 8(A\cos t + B\sin t) = \frac{65}{7}\sin t$$

$-A + 4B + 8A = \emptyset$ .... $B = 1,\ A = -4/7$

$-B - 4A + 8B = \frac{65}{7}$ (omitted)

$$x_p = -4/7\cos t + \sin t$$

(III) The general solution

$$x = e^{-2t}(c_1\cos 2t + c_2\sin 2t) - 4/7\cos t + \sin t$$

(IVP): $x(\emptyset) = 3/7$, $x'(\emptyset) = \emptyset$

$x(\emptyset) = 3/7 \Rightarrow e^0(c_1 \cdot 1 + c_2 \cdot 0) - 4/7 + \emptyset = 3/7$

$c_1 - 4/7 = 3/7 \Rightarrow c_1 = 1$

$$x' = -2e^{-2t}(c_1\cos 2t + c_2\sin 2t) + e^{-2t}(-2c_1\sin 2t + 2c_2\cos 2t) + 4/7\sin t + \cos t$$

$x'(0) = \emptyset \Rightarrow -2(c_1 + c_2 \cdot 0) + e^0(-2c_1 \cdot 0 + 2c_2 \cdot 1) + 0 + 1 = \emptyset$

$-2 + 2c_2 + 1 = 0$, $2c_2 = 1$ $c_2 = 1/2$

$$x = e^{-2t}(\cos 2t + 1/2\sin 2t) - 4/7\cos t + \sin t$$

is the solution.