3.5 Variation of Parameters

Oct. 23/17

Applied Anal.

To find a particular solution

"undetermined coeffs":

$ay'' + by' + cy = f(x)$

1. a, b, c are constants
2. $f(x)$ must be one of the "12 types" (table on p. 129)

$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$

(1) Solve $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

Say, the general solution is:

$y = c_1y_1 + c_2y_2$

(2) To find a particular solution y_p

3.2: Reduction of order $y_p = u(x)y_1$

$a_2(x)(uy_1)'' + a_2(x)(uy_1)' + a_0(x)(uy_1) = f(x)$

$y'' + P(x)y' + Q(x)y = f(x)$ - Standard Form

(1) Assume y_1, y_2 form a fundamental set of solutions for $y'' + P(x)y' + Q(x)y = 0$

$$\begin{cases} y_1'' + P(x)y_1' + Q(x)y_1 = 0 \\ y_2'' + P(x)y_2' + Q(x)y_2 = 0 \end{cases} - 1 \qquad y = c_1y_1 + c_2y_2$$

(2) Let $y_p = U_1(x)y_1 + U_2(x)y_2$ be a particular solution of the (*) for some functions $U_1(*)$, $U_2(*)$

$\to (u_1y_1 + u_2y_2)'' + P(x)(u_1y_1 + u_2y_2) + uQ(u_1y_1 + u_2y_2) \cdots$

$\cdots = f(x)$

Impose $y_1u_1' + y_2u_2' = 0$ - 2

$y_p' = \cancel{u_1'y_1} + u_1y_1' + \cancel{u_2'y_2} + u_2y_2' = u_1y_1' + U_2y_2'$

$Q(x)[\cancel{U_1y_1} + \cancel{U_2y_2}] = f(x)$

$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$

$(u_1'y_1' + \cancel{u_1y_1''} + U_2'y_2' + \cancel{u_2y_2''}] + P(x)[\cancel{u_1y_1'} + \cancel{U_2y_2'}] +$

$$u_1(y_1'' + P(x)y_1' + Q(x)y_1) = 0$$

$$u_1'y_1' + u_2'y_2' = f(x)$$

So we solve for $u_1(x)$, $u_2(x)$

$$\begin{cases} y_1u_1' + y_2u_2' = 0 & (\text{imposed}) \\ y_1'u_1' + y_2'u_2' = f(x) & (DE) \end{cases}$$

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}, \quad \text{where}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_2 & f(x) \end{vmatrix}$$

Ex. Solve $x^2y'' - 4xy' + 6y = 2x^{-1}$ given that $y = c_1x^2 + c_2x^3$ is the general solution for the associated homogeneous eqn $x^2y'' - 4xy' + 6y = 0$

Sol. To find a particular solution y_p

" The undetermined coefficients " Cannot be used

$y_1 = x^2 \quad y_2 = x^3$

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

Use our formula to find $u_1(x)$, $u_2(x)$

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \qquad x^2y'' - 4xy' + 6y = 2x^{-1}$$

$f(x) = ?$ Standard Form

$$y'' - \frac{4x}{x^2}y' + \frac{6}{x^2}y = 2x^{-1}\cdot x^{-2} = 2x^{-3}$$

$$W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^{-3} & 3x^2 \end{vmatrix} = -2x^{-3} \cdot x^3 = -2$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} x^2 & 0 \\ 2x & 2x^{-3} \end{vmatrix} = 2x^{-1}$$

$$U_1' = \frac{-2}{x^4} = -2x^{-4} \qquad U_1 = \int -2x^{-4}\,dx = \frac{2x^{-4+1}}{-4+1}$$

$$= \tfrac{2}{3} \cdot x^{-3}$$

$$U_2' = \frac{2x^{-1}}{x^4} = 2x^{-5} \qquad U_2 = \int 2x^{-5}\,dx = \frac{2\,x^{-5+1}}{-5+1}$$

$$= -\tfrac{2}{4}\,x^{-4}$$

$$= -\tfrac{1}{2}\,x^{-4}$$

$$y_p = U_1 y_1 + U_2 y_2 = (\tfrac{2}{3}x^{-3})(x^2) + (-\tfrac{1}{2}x^{-4})(x^3)$$

$$y_p = (\tfrac{2}{3})x^{-1} + (-\tfrac{1}{2})x^{-1} = (\tfrac{4}{6})x^{-1} - (\tfrac{3}{6})x^{-1}$$

$$= (\tfrac{1}{6})x^{-1}$$

(3) The general solution is

$$y = c_1 x^2 + c_2 x^2 + \tfrac{1}{6}x^{-1}$$

<u>Ex.</u> Find the general Solution of

$$y'' - 2y' + y = x^3 e^x$$

using Variation of parameters

<u>Solution</u> (1) Solve the associated homo. equ. $y'' - 2y' + y = 0 \quad (y = e^{mx})$

Auxiliary Eqn $\quad m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0 \Rightarrow m_1 = 1,\ m_2 = 1$$

$$y = c_1 e^x + c_2 x e^x$$

$$y_1 = e^x$$

$$y_2 = x e^x$$

$$y_p = u_1 y_1 + u_2 y_2 \quad , \quad u_1' = \frac{W_1}{W} \quad , \quad u_2' = \frac{W_2}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^x(e^x + \cancel{xe^x}) - \cancel{xe^{2x}}$$

$$= e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & xe^x \\ x^3 e^x & e^x + xe^x \end{vmatrix} = -x^4 e^{2x}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} e^x & 0 \\ e^x & x^3 e^x \end{vmatrix} = x^3 e^{2x}$$

$$u_1' = \frac{W_1}{W} = \frac{-x^4 \cancel{e^{2x}}}{\cancel{e^{2x}}} = -x^4$$

$$u_2' = \frac{W_2}{W} = \frac{x^3 \cancel{e^{2x}}}{\cancel{e^{2x}}} = x^3$$

$$u_1 = -x^5/5$$

$$u_2 = x^4/4$$

3.5 Variation of Parameters

Oct. 25/17

Applied Anal.

Standard Form $y'' + P(x)y' + Q(x)y = f(x)$

(1) Let the general solution of the associated homo. eq'n $y'' + P(x)y' + Q(x)y = 0$

$y = c_1 y_1 + c_2 y_2$

(2) Let $y_p = u_1 y_1 + u_2 y_2$ be a particular solution of the non-homo eq'n for functions u_1 and u_2.

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Higher-order Equations (3rd - order)

Standard Form $\boxed{y''' + P_2(x)y'' + P_1(x)y' + P_0(x)y = f(x)}$ $\boxed{(*) = f(x)}$

(1) Let $y = c_1 y_1 + c_2 y_2 + c_3 y_3$ be the general solution of the associated homo. eq'n $y''' + P_2(x)y'' + P_1(x)y' + P_0(x)y = 0$

(2) Assume $y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$ is a particular solution of $(*)$ for u_1, u_2, u_3 (functions).

$$\begin{cases} y_1 u_1' + y_2 u_2' + y_3 u_3' = 0 & \text{(imposed)} \\ y_1' u_1' + y_2' u_2' + y_3' u_3' = 0 & \\ y_1'' u_1' + y_2'' u_2' + y_3'' u_3' = f(x) & \text{(DE)} \end{cases}$$

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}, \quad u_3' = \frac{W_3}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix} \quad W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}$$

3.6 Cauchy-Euler Equations

n^{th} order:

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y = g(x)$$

where $a_0, a_1, \dots, a_n$ are constants.

Second order:

$$a x^2 y'' + b x y' + c y = g(x)$$

Solve the associated homo. eq'n.

$$a x^2 y'' + b x y' + c y = 0$$

$\boxed{\text{Try } y = x^m}$

$$a x^2 (x^m)'' + b x (x^m)' + c(x^m) = 0$$

$$a x^2 \cdot m(m-1) x^{m-2} + b x m x^{m-1} + c x^m = 0$$

$$a m(m-1) x^m + b m x^m + c x^m = 0$$

$$\boxed{a m(m-1) + b m + c = 0}$$

— Auxiliary equ. for Cauchy-Euler equation

(I) $m_1 \neq m_2$ are distinct real roots :

$$\boxed{y = c_1 x^{m_1} + c_2 x^{m_2}}$$

Ex. Solve $x^2 y'' + 5xy' - 5y = \emptyset$

Solution Cauchy - Euler equation !

($y = m^x$) Auxiliary equation for Cauchy - Euler :

$m^2 + (5-1)m - 5 = \emptyset$, $m^2 + 4m - 5 = \emptyset$

$(m+5)(m-1) = \emptyset$

$m_1 = 1$, $m_2 = -5$

$$y = c_1 x + c_2 x^{-5}$$

(II) $m_1 = m_2$ is a repeated real root :

then $y_1 = x^{m_1}$ is a first solution

(3.2) $$\boxed{y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx}$$

$$am^2 + (b-a)m + c = \emptyset$$

$$m_1 = \frac{-(b-a) + \sqrt{\emptyset}}{2a}$$

$$ax^2 y'' + bxy' + cy = \emptyset$$

Standard Form : $y'' + \frac{bx}{ax^2} y' + \frac{c}{ax^2} y = \emptyset$

$$P(x) = b/ax$$

$$y_2 = x^{m_1} \int \frac{e^{-\int b/ax \, dx}}{(x^{m_1})^2} dx$$

$$y_2 = x^{m_1} \int \frac{e^{-b/a \ln x}}{x^{2m_1}} dx \Rightarrow x^{m_1} \int \frac{(e^{\ln x})^{-b/a}}{x^{2m_1}} dx$$

$$= x^{m_1} \int \frac{x^{-b/a}}{x^{-b/a} \cdot x} \, dx$$

$$= x^{m_1} \int \frac{1}{x} \, dx \Rightarrow x^{m_1} \cdot \ln x$$

(II) $m_1 = m_2$ is a repeated real root

$$\boxed{y = C_1 x^{m_1} + C_2 x^{m_1} \ln x}$$

For higher-order equations,

if m_1 is a repeated real root, repeated

K times: $x^{m_1}, x^{m_1} \ln x, x^{m_1} (\ln x)^2, \ldots, x^{m_1} (\ln x)^{K-1}$

Ex. Solve $x^2 y'' + 5xy' + 4y = 0$

Solution: (Cauchy-Euler equation) $(y = x^m)$

$m^2 + (5-1)m + 4 = 0$

$m^2 + 4m + 4$

$(m+2)(m+2) \rightarrow m_1 = -2$

$y = C_1 x^{-2} + C_2 x^{-2} \ln x$

3.6 Cauchy-Euler Equations

Oct. 27/17

Applied Anal.

$ax^2y'' + bxy' + cy = 0$

$y = x^m$

Auxiliary Eqn $\quad am^2 + (b-a)m + c = 0$

(I) $m_1 \neq m_2$ are distinct real roots.

$y_1 = c_1 x^{m_1} + c_2 x^{m_2}$

(II) m_1 is a repeated real root

$y = c_1 x^{m_1} + c_2 x^{m_1} \ln x$

(III) $m_1 = \alpha + \beta i$, $m_2 = \alpha - \beta i$ are complex roots

$y = c_1 x^{\alpha} (\cos \beta \ln x) + c_2 x^{\alpha} (\sin \beta \ln x)$

Ex. Solve $x^2y'' + 5xy' + 5y = 0$

Solution Cauchy-Euler equation $\quad y = x^m$

Auxiliary eqn : $m^2 + (5-1)m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{4^2 - (4)(5)}}{2} = \frac{-4 \pm \sqrt{-4}}{2}$$

$$m = \frac{-4 \pm 2i}{2} \qquad m_1 = (-2) + i \qquad \alpha = -2$$

$$m_2 = (-2) - i \qquad \beta = 1$$

$y = c_1 x^{-2} \cos(\ln x) + c_2 x^{-2} \sin(\ln x)$

Ex. Solve $x^3y''' + 5x^2y'' + 7xy' + 8y = 0$

Solution Cauchy-Euler equation $\quad y = x^m$

$x^3(x^m)''' + 5x^2(x^m)'' + 7x(x^m)' + 8(x^m) = 0$

$\left((x^m)' = mx^{m-1}, \quad (x^m)'' = m(m-1)x^{m-2} \right.$

$\left. (x^m)''' = m(m-1)(m-2)x^{m-3} \right)$

$$x^3\left[m(m-1)(m-2)x^{m-3}\right] + 5x^2\left[m(m-1)x^{m-2}\right] + \cdots$$
$$\cdots\ x\left[mx^m\right] + 8x^m = 0$$
$$m(m-1)(m-2)x^m + 5m(m-1)x^m + 7mx^m + 8x^m = 0$$
$$m(m-1)(m-2) + 5m(m-1) + 7m + 8 = 0$$
$$m(m^2-3m+2) + 5m^2 - 5m + 7m + 8 = 0$$
$$m^3 - 3m^2 + 2m + 5m^2 + 2m + 8 = 0$$
$$m^3 + 2m^2 + 4m + 8 = 0, \qquad m = -2$$
(by inspection)

$$(m+2)(\ ?\ ) = 0$$

$$\begin{array}{r} m^2 \qquad + 4 \\ (m+2)\ \overline{)\ m^3 + 2m^2 + 4m + 8} \\ (-)\ m^3 + 2m^2 \qquad\qquad \\ \hline 0 + 0 + 4m + 8 \\ 4m + 8 \\ \hline \underline{\underline{R0}} \end{array}$$

$$\Rightarrow (m^2+4)(m+2) = 0$$
$$\& \quad m = \pm\sqrt{-4}$$
$$= m_1 = -2$$
$$m_2 = 0 + 2i \qquad \alpha = 0$$
$$m_3 = 0 - 2i \qquad \beta = 2$$

then...

$$x^\alpha \cos(\beta \ln x) + x^\alpha \sin(\beta \ln x)$$
$$x^0 \cos(\beta \ln x) + x^0 \sin(\beta \ln x)$$
$$\Rightarrow \underline{\cos(2\ln x) + \sin(2\ln x)}$$