OCT. 16/17

APPLIED ANAL.

Put List # on Cover page (2)

3.3 Homogeneous Linear DEs with Constant Coefficients

$ay'' + by' + cy = 0$

where a, b, c are constants

$\boxed{y = e^{mx}}$

Auxiliary Equation : $am^2 + bm + c = 0$

$m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(I) : Two real roots (distinct)

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

(II) : $m_1 = m_2$ (repeated root)

$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$

(III) $m_1 = \alpha + \beta i$, $m_2 = \alpha - \beta i$

$i = \sqrt{-1}$

$y = c_1 e^{\alpha(x)} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$

Ex. Solve : (1) $y'' + k^2 y = 0$

(2) $y'' - k^2 y = 0$

Solution (1) $y = e^{mx}$, Auxiliary Equation

$m^2 + k^2 = 0$; $m^2 = -k^2$

$m = \pm\sqrt{-k^2} = \pm\sqrt{k^2} \cdot \sqrt{-1} = \pm ki$

$m_1 = ki$; $m_2 = -ki$, $\alpha = 0$, $\beta = k$

$y = c_1 \cos(kx) + c_2 \sin(kx)$

(2) $y'' - k^2 y = 0$ $\boxed{y = e^{mx}}$

Auxiliary Eq'n : $m^2 - k^2 = 0$

$m^2 = k^2$, $m = \pm\sqrt{k^2} = \pm k$

$m_1 = k$, $m_2 = -k$

$y = c_1 e^{kx} + c_2 e^{-kx}$

Higher Order DE's

$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$

where $a_0, a_1, \dots, a_{n-1}, a_n$ are constants

$\boxed{y = e^{mx}}$ The Auxiliary Equation

$$\boxed{a_n m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0 = 0}$$

(I) If $m_1, m_2, \dots, m_n$ are n distinct roots,

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$

(II) If m_1 is a real root repeated k times

$e^{m_1 x}, \; x e^{m_1 x}, \; x^2 e^{m_1 x}, \; \dots, \; x^{k-1} e^{m_1 x}$

are k linearly indep. solutions.

(III) If $m_1 = \alpha + \beta i$ is a complex root repeated k times

then $m_2 = \alpha - \beta i$ is a root repeated k times

$e^{\alpha x}\cos(\beta x), \; x e^{\alpha x}\cos(\beta x), \; \dots, \; x^{k-1} e^{\alpha x}\cos(\beta x)$

$e^{\alpha x}\sin(\beta x), \; x e^{\alpha x}\sin(\beta x), \; \dots, \; x^{k-1} e^{\alpha x}\sin(\beta x)$

are $2k$ linearly indep. solutions.

Ex. Solve $y''' + 3y'' - 4y = 0$

Solution: $y = e^{mx}$ (Auxiliary Eqn)

$m^3 + 3m^2 - 4 = 0 \longrightarrow (1 + 3 - 4 = 0)$

By inspection, $m_1 = 1$ is a root.

$\iff$ $m - 1$ is a factor of $m^3 + 3m^2 - 4$

$m^3 + 3m^2 - 4 = (m-1)(\,?\,)$

$\dfrac{m^3 + 3m^2 - 4}{(m-1)} = (\,?\,)$

$m^2 \quad 4m \quad 4$

$(m-1)\;)\; m^3 + 3m^2 - 4$

$\ominus \; m^3 - m^2 - 0$

$0 \quad 4m^2 \quad - 4$

$\ominus \; 4m^2 - 4m$

$+4m - 4$

$4m - 4$

$4m - 4$

$R\,0$

then

$m^3 + 3m^2 - 4 = (m-1)(m^2 + 4m + 4)$

$(m-1)(m^2+4m+4)$

$m-1=0 \qquad (m+2)^2 = 0$

$m_1 = 1 \qquad m+2=0$

$m_2 = -2, \quad m_3 = -2$

$y = c_1 e^{1x} + C_2 e^{-2x} + C_3 x_3 e^{-2x}$ is the general solution

<u>Ex.</u> Solve $y^{(4)} + 2y^{(2)} + y = 0$

<u>Solution</u> $y = e^{mx}$ Auxiliary Eq'n.

$m^4 + 2m^2 + 1 = 0$

$(m^2+1)^2 = 0$

$(m^2+1) = 0$

$m^2 = -1, \quad m = \pm i$

$m_1 = i$ repeats two times, $\alpha = 0$, $\beta = 1$

$m_2 = -i$ repeats two times

$\cancel{e^{\alpha x}}\cos(x), \; x\cancel{e^{\alpha x}}\cos(x) \quad \Big\} \quad \cos(x), \; x\cos(x)$

$\cancel{e^{\alpha x}}\sin(x), \; x\cancel{e^{\alpha x}}\sin(x) \quad \Big\} \quad \sin(x), \; x\sin(x)$

The general solution

$y = c_1\cos x + C_2 x\cos x + C_3 \sin x + C_4 x \sin x$

<u>Ex</u> Solve: $y^{(6)} + y^{(5)} = 0$

<u>Solution</u>: $y = e^{mx}$ - Auxiliary Equation

Auxiliary Eq'n $m^6 + m^5 = 0$

$m^5(m+1) = 0$

$\underbrace{m \cdot m \cdot m \cdot m \cdot m}_{m=0 \text{ repeats 5 times}} (m+1) = 0$

$m_1 = m_2 = m_3 = m_4 = m_5 = 0 \qquad m_6 = -1.$

$y = c_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4 + C_6 e^{-x}$

Oct. 18/17
APPLIED ANAL.

Ex. Solve $y'' + 4y' + 4y = x^2 - 2x$

Solution (1) Solve the associated homo. eq'n

$$y'' + 4y' + 4y = 0 \quad , \quad y = e^{mx}$$

Auxiliary Eq'n : $m^2 + 4m + 4 = 0$

$(m+2)^2 = 0 \qquad (m_1+2) = 0 , \qquad (m_2+2) = 0$

$\qquad\qquad m_1 = -2 \qquad m_2 = -2$

$$y = C_1 e^{-2x} + C_2 x e^{-2x}$$

(2) To Find a particular Solution of the non-homogeneous eq'n y_p

From (3.2), "Reduction of Order"

$y_p = u(x)\, y$, For some $u(x)$

New method to Find y_p :

$y'' + 4y' + 4y = x^2 - 2x$

Try $y_p = Ax^2 + Bx + C$ For some A, B, C

$(Ax^2 + Bx + C)'' + 4(Ax^2 + Bx + C)' + 4(Ax^2 + Bx + C) = x^2 - 2x$

$(2A) + 4(2Ax + B) + 4(Ax^2 + Bx + C) = x^2 - 2x$

$\rightarrow 4Ax^2 + (8Ax + 4Bx) + (2A + 4B + 4C) = x^2 - 2x$

$\rightarrow 4Ax^2 + (8A + 4B)x + (2A + 4B + 4C) = x^2 - 2x$

$\longrightarrow 4A = 1$; then $\boxed{A = 1/4}$

$8A + 4B = -2 \Rightarrow 8(1/4) + 4B = -2$; then $\boxed{B = -1}$

$2A + 4B + 4C = 0 \Rightarrow 2(1/4) + 4(-1) + 4C = 0$; then $\boxed{C = 7/8}$

then $y_p = \frac{1}{4}x^2 - x + \frac{7}{8}$

(3) The general solution of the nonhomo. eq'n

$$y = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{1}{4}x^2 - x + \frac{7}{8}$$

Ex. Find a particular solution of

$y'' + 2y' + y = (2x-3) + (5xe^x)$

Solution assumption

$y_p = (Ax+B) + (Cx+D)e^x$

$y_p' = (A) + (C)e^x + (Cx+D)e^x = A + (Cx+D+C)e^x$

$y_p'' = Ce^x + (Cx+D+C)e^x = (Cx+D+2C)e^x$

$LHS = y_p'' + 2y_p' + y_p$

$= (Cx+D+2C)e^x + 2[A + (Cx+D+C)e^x] + [(Ax+B) + (Cx+D)e^x]$

$= RHS = 2x-3 + (5x)e^x$

$Ax + B + e^x[(C+2C+C)x + (D+2C+2D+2C+D)]$

$= 2x-3 + (5x+0)e^x$

$A = 2 \qquad 4C = 5 \rightarrow C = 5/4$

$B = -3 \qquad 4D + 4C = 0 \rightarrow D = -5/4$

$y_p = 2x - 3 + (5/4x - 5/4)e^x$

Ex. Solve $y'' + 3y' + 2y = 2e^{-x}$

Solution (1) Solve the associated homo eq'n.

$y'' + 3y' + 2y = 0 \; ; \; y = e^{mx}$

Auxiliary equation : $m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0 \; ; \; m_1 = -1, \; m_2 = -2$

$y = c_1 e^{-x} + c_2 e^{-2x}$

(2) Find a particular solution y_p :

Assume $y_p = Ae^{-x}$ (normally)

$(Ae^{-x})'' + 3(Ae^{-x})' + 2(Ae^{-x}) = 2e^{-x}$

$Ae^{-x} - 3Ae^{-x} + 2Ae^{-x} = 2e^{-x}$

$(A - 3A + 2A)e^{-x} = 2e^{-x}$

$0e^x = 2e^{-x}$??

Why? Assumed $y_p = Ae^{-x}$

Revise! New assumption $y_p = Axe^{-x}$

$Ae^{-x} \rightarrow x(Ae^{-x}) = Axe^{-x}$

MATH 2050 MIDTERM TEST

OCT. 20/17
APPLIED ANAL.

Friday, October 27TH /17
4:30 → 5:30 PM AT 1003

1. Close book except 1/2 page of hand-written notes, one side only
2. No calculator may be used
3. Chapter: 2.2, 2.3, 2.4
   2.7, 3.1, 3.2
4. Put your cell phone away

Cont'd: Find a particular solution

$y_p' = Ae^{-x} + Ax(-e^{-x}) = A(1-x)e^{-x}$

$y_p = (Ae^{-x})^x$ (revised)

$y_p'' = -Ae^{-x} + A(1-x)(-e^{-x}) = A(x-2)e^{-x}$

$y_p'' + 3y_p' + 2y_p = 2e^{-x}$

$A(x-2)e^{-x} + 3(-Ae^{-x})^x + 2(Ae^{-x})^x$

$y_p = 2xe^{-x}$

(3) The general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + 2xe^{-x}$$

Table on page 129

<u>Rule</u> Let y_c be the general solution of the associated homo. eq'n

$$ay'' + by' + cy = 0$$

(I) If the assumed y_p is not in y_c, then keep this y_p

(II) If the assumed y_p is in the y_c, then we take a new assumption $x y_p$

If, again, $x y_p$ is in y_c, we revise it, take $x(x y_p) = x^2 y_p$, ...

Ex. Solve $y'' - 4y' + 2y = e^{2x}$

Solution (1) Solve $y'' - 4y' + 4y = 0$ $(y = e^{mx})$

Auxiliary eq'n $m^2 - 4m + 4 = 0$

$(m-2)(m-2) \Rightarrow (m-2)^2$ $(m_1 = m_2 = 2)$

$y = c_1 e^{2x} + c_2 x e^{2x}$

(2) To find a particular solution y_p, $y_p = Ae^{2x}$

(Revise) $y_p = Axe^{2x} \longrightarrow y_p = Ax^2 e^{2x}$

$c_1 = 0$, $c_2 = A$

$y_p = Ax^2 e^{2x}$

$(Ax^2e^{2x})'' - 4(Ax^2e^{2x})' + 4(Ax^2e^{2x}) = e^{2x}$

$\Rightarrow [2Ae^{2x}(2x^2 + 4x + 1)] - 4[2A(x^2 + x)e^{2x}] + 4(Ax^2e^{2x}) = e^{2x}$

$Ae^{2x}[2(2x^2 + 4x + 1) - 8(x^2 + x) + 4x^2] = e^{2x}$

$Ae^{2x}[(4 - 8 + 4)x^2 + (8 - 8)x + 2] = e^{2x}$

$2A e^{2x} = e^{2x}$; $2A = 1 \Rightarrow A = \frac{1}{2}$

$y_p = \frac{1}{2} x^2 e^{2x}$

(3) The general solution

$y = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{2} x^2 e^{2x}$

Ex. Solve $y'' + y = 4x + 10\sin x$

Solution: (1) Associated homo eq'n:

$y'' + y = 0$, $y = e^{mx}$, Auxiliary Eq'n

$m^2 + 1 = 0$, $m \pm \sqrt{-1} = \pm i$ $\quad \alpha = 0$, $\beta = 1$

$y = C_1 \cos(x) + C_2 \sin(x)$ $\quad \boxed{e^{\alpha x} = e^{0x} = 1}$

(2) To Find a particular Solution y_p

$$y_p = \underbrace{(Ax + B)}_{\text{in } y_c?} + (C\cos x + D\sin x)$$

Check it → "type" by "type"

Revise it, new assumption

$y_p = (Ax + B) + x(C\cos x + D\sin x)$

$y_p = (Ax + B) + (Cx\cos x + Dx\sin x)$

⋮

$A = 4$, $B = 0$, $C = -5$, $D = 0$

$y_p = 4x - 5x\cos x$

(3) The general Solution is

$$y = C_1\cos x + C_2\sin x + 4x - 5x\cos x$$

Higher-order equations

Ex. Determine the form of a particular solution

$y^{(4)} + y''' = 1 - e^{-x}$

Solution: (1) Solve the associated homo. eq'n.

$y^{(4)} + y''' = 0$, $y = e^{mx}$

Auxiliary Eq'n: $m^4 + m^3 = 0 \Rightarrow m^3(m+1) = 0$

↳ $m \cdot m \cdot m(m+1) \Rightarrow m_1 = m_2 = m_3 = 0$, $m_4 = -1$

$y = C_1 + C_2 x + C_3 x^2 + C_4 e^{-x}$

The form of a particular solution

$y_p = \underset{Ax,\ Ax^2}{A} + Be^{-x}$ (From the table) → $y_p = Ax^3 + Bxe^{-x}$