Oct. 6/17
Applied Anal

$xu'' + 6u' = x^{-4}$ (second order)

Sub. : $w = u'$; $w' = u''$

$xw' + 6w = x^{-4}$ (first order)

$w' + 6/x\, w = x^{-5}$

An integrating factor : $e^{\int 6/x\, dx} = e^{6 \ln|x|} = (e^{\ln|x|})^6 = x^6$

$d/dx\,(x^6 \cdot w) = (x^6)(x^{-5}) = x$

$x^6 \cdot w = \int x\, dx = x^2/2$

$u'w = 1/2\, x^2 \cdot x^6 = 1/2\, x^{-4}$

$u = \int 1/2\, x^{-4}\, dx = (1/2)\dfrac{x^{-4+1}}{-4+1} = -1/6\, x^{-6}$

$y_p = uy_1 = (-1/6\, x^{-3}) \cdot (x^4) = -1/6\, x$

$y_p = uy_1 = (-1/6\, x^{-3}) \cdot (x^4) = -1/6\, x$ (I don't know why I wrote this twice)

(3) The general solution for the nonhomo. eq'n

$y = c_1 x^4 + c_2 x^{-1} - 1/6\, x$

3.3 Homogeneous Linear DE's with constant coefficients

$2y'' + 3y' - 5y = 0$ ✓ (coefficient constant)

$x^2 y'' + e^x y' + y = 0$ × (coefficient not constant)

$\boxed{ay'' + by' + cy = 0}$ where a, b, c are constants

Need a fundamental set of solutions (two linearly indep. solutions y_1, y_2)

$\boxed{\text{Try } y = e^{mx}}$

$a(e^{mx})'' + b(e^{mx})' + c(e^{mx}) = 0$

- Solve it for a constant m

$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$

$(am^2 + bm + c)e^{mx} = \emptyset \quad (\div e^{mx})$

$\boxed{am^2 + bm + c = \emptyset}$ (Auxiliary equation)

$m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (I) $m_1 \neq m_2$ two distinct real roots

$(b^2 - 4ac > \emptyset)$

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Ex. Solve $y'' - 3y' + 2y = \emptyset$

Solution: A homogeneous linear equation w/ const. coefficient

$y = e^{mx}$ (Auxiliary equation)

$m^2 + (-3)m + 2 = \emptyset$

$(m-1)(m-2) = \emptyset$

$m = 1, \quad m = 2$

The general solution: $\boxed{y = c_1 e^{x} + c_2 e^{2x}}$

(II) $m_1 = m_2$ is a repeated real root

$m_1 = m_2 = -b/2a \quad (b^2 - 4ac = \emptyset)$

$y_1 = e^{m_1 x} = e^{-\frac{b}{2a}x}$ is a solution.

How to get a second one y_2?

$y_2 = y_1 \displaystyle\int \frac{e^{-\int P(x)dx}}{y_1^2} dx \qquad P(x) = ?$

Standard form: $y'' + b/a\, y' + c/a\, y = \emptyset$

$y_2 = e^{m_1 x} \displaystyle\int \frac{e^{-\int b/a\, dx}}{(e^{-\frac{b}{2a}x})^2} dx$ where $(e^{\alpha})^{\beta} = e^{\alpha \cdot \beta}$

$y_2 = e^{m_1 x} \displaystyle\int \frac{e^{-b/a\, x}}{e^{-b/a\, x}} dx = e^{m_1 x} \int 1\, dx$

$= e^{m_1 x} \cdot x$

The general solution

$y_1 = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

<u>Ex:</u> Solve $y'' + 6y' + 9y = 0$

<u>Solution:</u> Homo-linear eq'n with constant coeffs

$y = e^{mx}$

Auxiliary Eq'n: $m^2 + 6m + 9 = 0$

$(m+3)^2 = 0 \Rightarrow (m+3)(m+3) = 0$

$m_1 = m_2 = -3$

The general solution is

$y = c_1 e^{-3x} + c_2 x e^{-3x}$

(III) Complex roots $m_1 = \alpha + \beta i$

$m_2 = \alpha - \beta i$

$(b^2 - 4ac < 0)$

$am^2 + bm + c = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{(4ac - b^2)(-1)}}{2a}$$

$$= \frac{-b \pm \sqrt{4ac - b^2} \cdot \sqrt{-1}}{2a}$$

$$= (-b/2a) \pm \left(\frac{\sqrt{4ac - b^2}}{2a}\right)(\sqrt{-1})$$

$\rightarrow (\sqrt{-1}) = i$

<u>Ex.</u> Solve $y'' + y' + y = 0$

<u>Solution:</u> $y = e^{mx}$ (Auxiliary Eq'n)

$m^2 + m + 1 = 0$