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Applied Anal

Homogeneous Equation

$$a_{(n)}x\, y^{(n)} + a_{(n-1)}(x)\, y^{(n-1)} + \dots + a_1(x)\, y' + a_0(x)\, y = 0$$

The general solution

$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$

where $\{y_1, y_2, \dots, y_n\}$ is a fundamental set of solutions, ie.

(1) n solutions $y_1, y_2, \dots, y_n$

(2) $y_1, y_2, \dots, y_n$ are linearly indep.

$(\Leftrightarrow \; W(y_1, y_2, \dots, y_n) \neq 0)$

Nonhomogeneous equation

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \dots + a_1(x)\, y' + a_0(x)\, y^{\circ} = g(x) \quad (K)$$

<u>Thm 3.6</u> If y_p is a particular solution of (K) and $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is the general solution of the associated homo eqn:

$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n + y_e$

<u>Ex.</u> Solve $y''' - 5y'' + 2y' + 8y = 8x^2 + 4x - 16$

<u>Solution</u>: (1) need a particular solution y_p: Given that $y_p = x^2$ is a part. solution

(2) need to solve the <u>associated homo. eq'n</u>: $y''' - 5y'' + 2y' + 8y = 0$

Given that: $y_1 = e^{-x}$, $y_2 = e^{2x}$, $y_3 = -e^{4x}$ are linearly indep. solutions

$y = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{4x}$

(3) The general solution of the homo. eq'n.

$y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{4x} + x^2$

Example

Verify that $y = c_1 e^{-x} + c_2 e^{3x} - 4/3 x + 23/9$
is the general solution of: $y'' - 2y' - 3y = 4x - 5$

Solution

(1) Verify that: $y_p = -4/3 + 23/9$
is a particular solution of: $y'' - 2y' - 3y = 4x - 5$

(2) Verify that: $y = c_1 e^{-x} + c_2 e^{3x}$
is the general solution for the associated homo eq'n:
$y'' - 2y' - 3y = \emptyset$ i.e. both $y_1 = e^{x}$, $y_2 = e^{3x}$ are
linearly independent solutions of $y'' - 2y' - 3y = \emptyset$

(1) $y_p' = -4/3$, $y_p'' = \emptyset$

$= y_p'' - 2y_p' - 3y_p = \emptyset - 2(-4/3) - 3(-4/3 x + 23/9)$

$= 8/3 + 4x - 23/3 = 4x - 5$

RHS $= 4x - 5$. So $y_p = -4/3 x + 23/9$ is a solution

(2) $y_1 = e^{-x}$, $y_2 = e^{3x}$ are two solutions
of $y'' - 2y' - 3y = \emptyset$:

LHS $= y_1'' - 2y_1' - 3y_1 = e^{-x} - 2(-e^{-x}) - 3e^{-x} = \emptyset =$ RHS

RHS $= y_2'' - 3y_2 = 9e^{3x} - 2(3e^{3x}) - 3e^{3x} = \emptyset =$ RHS

$$W(e^{-x}, e^{3x}) = \begin{vmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{vmatrix} = 3e^{3x} \cdot e^{-x} - e^{3x}(e^{-x})$$
$$= 3e^{2x} + e^{2x}$$
$$= 4e^{2x} \neq \emptyset$$

$\therefore e^{-x}, e^{3x}$ are linearly indep.

(3) So $y = c_1 e^{-x} + c_2 e^{3x} + -4/3 x + 23/9$
is the general solution of the original eq'n

3.2 Reduction of Order (homo.)

$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$

If $y_1 \neq 0$ is a solution, how to find a second solution y_2 such that $\{y_1, y_2\}$ form a fundamental set of solutions.

[y_1 is a solution, cy_1 is a solution for any constant, but y_1, cy_1 are linearly dependent. $u(x)\, y_1 = y_2$?]

Ex. Given that $y_1 = x^3$ is a solution of $x^2y'' - 6y = 0$. Find a second solution y_2 on $(0, \infty)$ such that y_1, y_2 form a fundamental set of solutions.

Solution : Find a function $u(x)$ such that

$y_2 = u(x)\, y_1$ is a solution. $u(x) = ?$

$x^2(uy_1)'' - 6(uy_1) = 0$ (solve it for $u(x)$)

$y_2' = (uy_1)' = (u \cdot x^3)' = u'x^3 + u \cdot 3x^2$

$y_2'' = (uy_1)'' = u''x^3 + u'3x^2 + u'3x^2 + u6x$

$= u''x^3 + 6x^2u' + 6xu$

$x^2\left[u''x^3 + 6x^2u' + \cancel{6xu}\right] - \cancel{6ux^3} = 0$ [Eq'n]

$x^2(u''x^3 + 6x^2u') = 0 \qquad x^4(u''x + 6u')$

$xu'' + 6u' = 0$, [Substitution] $w = u'$

$xw' + 6w = 0 \qquad w' = u''$

First-order linear eq'n.

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Applied Anal

3.2 Reduction of Order (cont'd)

$w' + {}^6/_x w = 0$ $\qquad d/dx\,(e^{\int 6/x\,dx}\cdot w) = 0\cdot e^{\int 6/x\,dx}$

$= 0$

$e^{6\ln x}\cdot w = C$ $\qquad w = Cx^{-6}$

$u' = Cx^{-6}$ $\qquad u = \dfrac{Cx^{-6+1}}{-6+1} + C_1$

$u(x) = -\dfrac{x^5}{5}$ $\qquad y_2 = {}^1/_5 x^{-5}\cdot x^3 = \dfrac{-x^{-2}}{5}$

$e^{6\ln x} = (e^{\ln x})^6 = x^6$ $\qquad u = \int Cx^{-6}dx$

$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ $\quad [\sim \div a_2(x)]$

Standard form

$y'' + P(x)y' + Q(x)y = 0$

Let $y_1 \neq 0$ be a solution

Let $y_2 = u(x)y_1$ for some $u(x)$

$$\boxed{(uy_1)'' + P(x)(uy_1)' + Q(x)(uy_1) = 0}$$

Solve it for $u(x)$

$(uy_1)' = u'y_1 + uy_1'$

$(uy_1)'' = u''y_1 + u'y_1' + u'y_1' + u'y_1''$

$= u''y_1 + 2u'y_1' + uy_1''$

$$[u''y_1 + 2u'y_1' + \cancel{uy_1''}] + P(x)[u'y_1 + \cancel{uy_1'}] + Q(\cancel{uy_1}) = 0$$

$u(y_1'' + P(x)y_1' + Q(x)y_1)$

$y_1u'' + P(x)y_1u' + 2u'y_1' = 0$

$y_1u'' + [P(x)y_1 + 2y_1']u' = 0$

Substitution; $w = u'$ $\quad w' = u''$

$y_1w' + [P(x)y_1 + 2y_1']w = 0$ – First order

$w' + (P(x) + {}^2/_{y_1}\cdot y_1')w = 0$

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$$\frac{d}{dx}\left(e^{\int P(x) + 2y_1'/y_1 \, dx} \cdot W\right) = 0$$

$$e^{\int P(x)dx + 2\ln|y_1|} \cdot W = C$$ (y_1 is a Function of x)

$$e^{\int P(x)dx} \cdot \cancel{e}^{\cancel{\ln} y_1^2} \cdot W = C$$

$$e^{\int P(x)dx} \cdot y_1^2 \cdot W = C$$

$$\hookrightarrow \quad W = \frac{1}{y_1^2} \cdot e^{-\int P(x)dx}, \quad u' = \frac{1}{y_1^2} e^{-P(x)dx}$$

$$\boxed{u(x) = \int \frac{e^{-\int P(x)dx}}{y_1^2} dx}$$

$$\boxed{y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx}$$

Ex. Given that $y_1 = x^3$ is a solution of

$x^2 y'' - 6y = 0$

Find the general solution

$\boxed{P(x) = 0}$

Solution:

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$= x^3 \int \frac{e^{-\int 0 dx}}{(x^3)^2}$$

$$y_2 = x^3 \int \frac{e^{-c}}{x^6} dx = x^3 \int \frac{1}{x^6} dx$$

$$\Rightarrow x^3 \cdot \frac{x^{-6+1}}{-6+1} = -1/5\, x^{-2}$$

The general solution:

$$y = C_1 x^3 + C_2 (-1/5\, x^{-2})$$

OR

$$y = C_1 x^3 + C_2 x^{-2}$$

Ex. Given that $y_1 = x^4$ is a solution
$x^2y'' - 2xy' - 4xy = 0$
Find the general solution.

Solution $y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}$ $\quad P(x) = ?$

$y'' - \frac{2x}{x^2}y' - \frac{4xy}{x^2} = 0$ $\quad P(x) = \frac{-2}{x}$ (don't miss the sign)

$y'' - \frac{2}{x}y' - \frac{4}{x}y = 0$ $\quad e^{\int 2/x\,dx} = e^{2\ln|x|} = \cancel{e}^{\ln x^2} = x^2$

$y_2 = x^4 \int \frac{e^{-\int -2/x\,dx}}{(x^4)^2}\,dx$

$= x^4 \int \frac{x^2}{x^8}\,dx$

$\Rightarrow x^4 \int x^{-6}\,dx = x^4 \cdot \left(\frac{x^{-6+1}}{-6+1}\right)$

$= -1/5\, x^{-1}$ or $y_2 = x^{-1}$

The general solution
$y = c_1x^4 + c_2x^{-1}$

Ex. Use the example above, solve
$x^2y'' - 2xy' - 4y = x$

Solution (1) Solve the asscociated homo. eq'n.
$x^2y'' - 2xy' - 4y = 0$
From the example above: $y = c_1x^4 + c_2x^{-1}$

(2) To Find a particular Solution, y_p,
let $y_p = u(x)y_1$, For some $u(x)$, $y_1 = x^4$
$x^2(uy_1)'' - 2x(uy_1)' - 4(uy_1) = x$ (Solve it For $u(x)$)

$(uy_1)' = u'y_1 + uy_1'$
$(uy_1)'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$
$= u''y_1 + 2u'y_1' + uy_1''$

$x^2[u''y_1 + 2u'y_1' + \cancel{uy_1''}] - 2x[u'y_1 + \cancel{uy_1'}] - 4[\cancel{uy_1}]$
$u(x^2y_1'' - 2xy_1' - 4y_1)$

$x^2y_1 u'' + 2x^2y_1' - 2xy_1u' = x$, $\boxed{\begin{matrix} y_1 = x^4 \\ y_1' = 4x^3 \end{matrix}}$
$x^2 \cdot x^4 \cdot u'' + (2x^2 \cdot 4x^3 - 2x \cdot x^4)u' = x$
$x^6 u'' + 6x^5 u' = x$
$xu'' + 6u' = x \cdot 1/x^5$
$= 1/x^4$

$xu'' + 6u' = x^{-4}$ Reduction of order
Sub. $w = u'$, $w' = u''$