Sept. 25/17
Applied Anal

<u>Example</u> : The rate of bacteria growth is proportional to the number of bacteria $N(t)$ present. We know that $N(0) = 1000$, and $N(1) = 5/4\, N(0)$. Find the time t at which the number of bacteria is doubled.

<u>Solution</u> : $\frac{dN}{dt} = kN$, $N(0) = 1000$, $N(1) = 5/4\, N(0)$

(1) $dN/dt - kN = 0$, $e^{\int -k\,dt} = e^{-kt}$

$$d/dx\,(e^{-k} \cdot N) = e^{-kt} \cdot 0 = 0$$
$$e^{-kt} \cdot N = \int 0\,dt = C, \quad N(t) = Ce^{kt}$$

(2) To find C and k : $N(0) = 1000$

$Ce^{k \cdot 0} = 1000$, $C = 1000$, $N(t) = 1000e^{kt}$

$N(1) = 5/4\,(1000) \Rightarrow 1000e^{k \cdot 1} = 5/4 \times 1000$

$e^{k} = 5/4$, $\ln e^{k} = \ln(5/4)$, $k = \ln 5/4$

$N(t) = 1000\, e^{t \ln(5/4)}$

(3) Find the time t such that $N(t) = 2000$

$1000\, e^{t \ln(5/4)} = 2000$

$e^{t \ln(5/4)} = 2$, $\ln e^{t \ln(5/4)} = \ln 2$

$t \ln(5/4) = \ln 2$

$$t = \boxed{\frac{\ln 2}{\ln(5/4)}}$$

Example:

$x(t)$ - radioactive substance: substance at time t, and $x(0) = x_0$. After 2 hours, $x(t)$ decreased by 2%. If the rate of decay is proportional to $x(t)$, find the half-life of the radioactive substance.

Solution: $dx/dt = kx$, $x(0) = x_0$, $\boxed{x(2) = x_0 - x_0 \cdot 2\%}$

$x(2) = 0.98\, x_0$

Find the time t (half-life) at which

$$x(t) = \tfrac{1}{2} x_0$$

(1) $dx/dt = kx$, $dx/dt - kx = 0$

$$d/dt\,(e^{\int k\,dt} \cdot x) = e^{\int k\,dt} \cdot 0 = 0$$

$$e^{-kt} \cdot x = c \; ; \quad x(t) = ce^{kt}$$

(2) Determine c and k:

$$x(0) = x_0 \Rightarrow ce^{k \cdot 0} = x_0 \qquad c = x_0$$

$$x(t) = x_0 e^{kt}$$

$$x(2) = 0.98\, x_0 \Rightarrow x_0 e^{k2} = 0.98\, x_0$$

$$e^{2k} = 0.98, \quad \ln e^{2k} = \ln 0.98, \quad 2k = \ln 0.98$$

$$k = \tfrac{1}{2} \ln(0.98) \; ; \quad x(t) = x_0 e^{t/2 (0.98)}$$

(3) Find the half-life t_h; i.e. $x(t_h) = \tfrac{1}{2} x_0$

$$x_0 e^{t_h/2 \ln(0.98)} = \tfrac{1}{2} x_0 \; ;$$

$$e^{t_h \cdot \frac{1}{2} \ln(0.98)} = \tfrac{1}{2} \; ; \quad \ln e^{t_h \cdot \frac{1}{2} \ln 0.98} = \ln(\tfrac{1}{2})$$

$$t_h \cdot \tfrac{1}{2} \ln(0.98) = \ln(\tfrac{1}{2}) \; ; \quad t_h = \frac{2 \ln \frac{1}{2}}{\ln 0.98}$$

Carbon dating: to determine the age of a fossil.
Known: half-life of C_{14} = 5600 years

Example

A fossilized bone is found to contain $\frac{1}{1000}$ the original amount of C_{14}. Determine the age of the fossil.

Solution $A(t)$ - amount of C_{14} in the bone

$$\frac{dA}{dt} = KA, \quad A(5600) = \tfrac{1}{2}A_0; \quad A(0) = A_0$$

$$\boxed{\text{Find the time } t \text{ at which } A(t) = \tfrac{1}{1000}A_0}$$

(1) $dA/dt = KA$, $\quad d/dt\,(e^{\int -k\,dt} \cdot A) = e^{\int -k\,dt} \cdot 0 = 0$

$e^{-kt} \cdot A = C$; $\boxed{A = Ce^{kt}}$

(2) Determine C and k

$A(0) = A_0 \Rightarrow Ce^{k \cdot 0} = A_0 \Rightarrow C = A_0$

$A(t) = A_0 e^{kt}$

$A(5600) = \tfrac{1}{2}A_0 \Rightarrow A_0 e^{k \cdot 5600} = \tfrac{1}{2}A_0$

$e^{5600k} = \tfrac{1}{2}$, $\ln e^{5600k} = \ln \tfrac{1}{2}$, $5600k = \ln \tfrac{1}{2}$

$k = \tfrac{1}{5600} \ln(\tfrac{1}{2})$

$$\boxed{A(t) = A_0 e^{t(1/5600)\ln 1/2}}$$

(3) age t: $A(t) = \frac{1}{1000}A_0$

$\cancel{A_0} e^{t(1/5000)\ln 1/2} = \tfrac{1}{1000}\cancel{A_0}$

$e^{t(1/5000)\ln 1/2} = \tfrac{1}{1000} \Rightarrow \ln e^{t(1/5000)\ln 1/2} = \ln \tfrac{1}{1000}$

$t \cdot \tfrac{1}{5600}\ln(\tfrac{1}{2}) = \ln(\tfrac{1}{1000})$

$$\boxed{t = 5600 \cdot \frac{\ln(1/1000)}{\ln(1/2)}}$$

Newton's Law of Cooling:

$T(t)$ - temperature of body

T_m - Surrounding medium

$dT/dt = K(T - T_m)$.

Example: A thermometer of reading 70°F is removed to outside where temperature is 10°F. After ½ minutes the temperature of the thermometer reads 50°F. What is the reading at $t = 1$? How long will it take to read 10°F?

Solution $dT/dt = k(T - T_m)$

$T_m = 10$, $T(0) = 70$, $T(1/2) = 50$

Find $T(1) = ?$ Find the time t such that $T(t) = 10$

(1) $dT/dt = k(T - 10)$, $dT/dt - kT = -10k$

$d/dt\,(e^{\int P(t)dt} \cdot T) = e^{\int P(t)dt} \cdot f(t)$

$d/dt\,(e^{\int -k dt} \cdot T) = e^{\int -k dt} \cdot (-10k)$

$e^{kt} \cdot (e^{-kt} \cdot T) = \int -10k\, e^{-kt}\, dt$

$= (10 e^{-kt} + c) \cdot e^{kt}$

$T = 10 + c e^{kt}$

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From previous example:

Solution: (1) $dT/dt = k(T-10)$, $dT/dt + (-k)T = -10k$

$d/dt\,(e^{\int -k\,dt} \cdot T) = e^{-\int k\,dt}$ ... (see previous)

(2) Find c and k : $T(t) = 10 + ce^{kt}$

$T(0) = 70 \Rightarrow 10 + ce^{k \cdot 0} = 70$

$10 + c = 70$ $\quad c = 60$, $\quad T(t) = 10 + 60e^{kt}$

$T(1/2) = 50 \Rightarrow 10 + 60e^{k \cdot 1/2} = 50$

$60e^{k/2} = 40$, $\quad e^{k/2} = 40/60 = 2/3$

$\ln e^{k/2} = \ln(2/3)$, $\quad k/2 = \ln(2/3)$, $\quad k = 2\ln(2/3)$

$$\boxed{T(t) = 10 + 60e^{2t\ln(2/3)}}$$

(3) $T(1) = 10 + 60e^{2\ln(2/3)} = 10 + 60(e^{\ln(2/3)})^2$

$= 10 + 60(2/3)^2$

When $T(t) = 10$? $\quad 10 + 60e^{2t\ln(2/3)} = 10$

$60e^{2t\ln(2/3)} = 0$, $\quad \boxed{e^{2t\ln(2/3)} = 0}$

$\ln(2/3) < 0 \Rightarrow e^{2t\ln(2/3)}$, decreases to 0 as $t \to \infty$

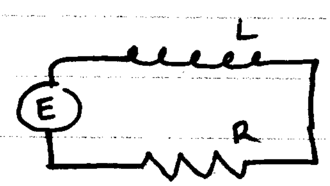
$$\lim_{t\to\infty} e^{2t\ln(2/3)} = 0$$

$$\lim_{t\to\infty} \left(1/e^{-2\ln(2/3)}\right)^t = 0$$

Series Circuits

Kirchhoff's Law:

$L(di/dt) + Ri = E(t)$



$i(t)$ = current

A 12v battery connected to series circuit. Inductance is $1/2$ henry, resistance 10 ohms. Determine the current i if the initial current is 0.

Solution: $E = 12$, $L = 1/2$, $R = 10$, $i(0) = 0$

$$\boxed{\frac{1}{2}\frac{di}{dt} + 10i = 12} \qquad i(0) = 0$$

(1) Standard Form $di/dt + 20i = 24$

$$\boxed{dy/dx + P(x)y = f(x)}$$

$$d/dx\left(e^{\int 20\,dt} \cdot i\right) = e^{\int 20\,dt} \cdot 24$$

$$\boxed{e^{20t} \cdot i} = \int 24 e^{20t}\,dt \implies 24\int e^{u} \cdot 1/20\,du$$

$u = 20t$

$du = 20\,dt$ then

$$= \frac{24}{20} e^{u} + C \Rightarrow \boxed{\frac{6}{5} e^{20t} + C}$$

$1/e^{20t}$ or e^{-20t}

$$\Rightarrow i(t) = 6/5 + ce^{-20t}$$

(2) $i(0) = 0 \Rightarrow 6/5 + C \cdot e^{-20(0)} = 0$

$6/5 + C = 0$; $C = -6/5$

$$\boxed{i(t) = 6/5 - 6/5 e^{-20t}}$$

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3. Higher-order DE's

Example Solve $x^2y'' + xy' + y = 0$

How to Find all of the solutions?

3.1 Linear DE's: Basic Theory

(1) Initial-Value Problem

$$a_n(x)y^{(n)} + a_n(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

$$y(x_0) = y_0;\ y'(x_0) = y_1;\ y''(x_0) = y_2;\ \cdots;\ y^{(n-1)}(x_0) = y_{n-1}$$

Where $y_0, y_1, \ldots y_{n-1}$ are constants.

Does the given IVP have a solution?

Thm. 3.1 If $a_n(x), a_{n-1}(x), \cdots, a_0(x)$ and $g(x)$ are continuous, and $a_n(x) \neq 0$, for every x in an interval I, then the IVP has a unique solution.

Ex. Solve the IVP

$$y''' + xy'' - y' + 2y = 0$$
$$y(1) = 0 \ ; \ y'(1) = 0 \ ; \ y''(1) = 0$$

Solution : 3^{rd}-order linear equation

From thm 3.1, this IVP has a unique solution.

By inspection, $y = 0$ is the solution

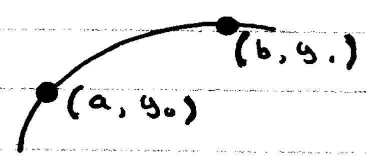
Boundary - value problem

Second - order

Solve $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

Subject to $y(a) = y_0$; $y(b) = y_1$

Ex. Solve the boundary value problem:



$$y'' = 2x, \quad y(1) = -1, \quad y(2) = 3$$

Solutions : $(y')' = 2x$, $y' = \int 2x\,dx$

$y' = x^2 + C_1$, $y = \int x^2 + C_1 \, dx$

$y = 1/3\, x^3 + C_1 x + C_2$

$\hookrightarrow$ Family of solutions

$y(1) = -1$ and $y(2) = 3$

$$\begin{cases} 1/3 + C_1 + C_2 = -1 \\ 1/3\,(2^3) + C_1 \cdot 2 + C_2 = 3 \end{cases} \rightarrow \begin{cases} 1/3 + C_1 + C_2 = -1 \\ 8/3 + 2C_1 + C_2 = 3 \end{cases}$$

$$\begin{cases} C_1 + C_2 = -1 - 1/3 = -4/3 & -(1) \\ 2C_1 + C_2 = 3 - 8/3 = 1/3 & -(2) \end{cases}$$

$(2) - (1) \Rightarrow \quad C_1 = 1/3 - (-4/3) = 5/3$

From (1), $C_2 = -4/3 - C_1 = -4/3 - 5/8$

$= -9/3 = -3$

$$\boxed{y = 1/3\, x^3 + 5/3\, x - 3}$$

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Applied An

3.1 Linear Equations : Basic Theory

$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0$
is called the associated homogeneous eq'n of

$$\boxed{a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots a_1(x)y' + a_0(x)y = g(x)}$$

<u>Ex.</u> $x^2y'' + xy' + y = \sin x + x^2$
The associated homogeneous eq'n:
$x^2y'' + xy' + y = 0$

<u>Thm. 3.2</u> (Superposition Principle - homo.)
Let $y_1, y_2, \ldots, y_k$ be solutions of the homo. eq'n: $a_n(x)y^{(n)} + \ldots + a_0(x)y = 0$
Then any linear combination
$$y = c_1y_1 + c_2y_2 + \ldots + c_ky_k$$
is also a solution to the homo eq'n.

<u>Goal</u>: Find the <u>least</u> <u>number</u> <u>of</u> <u>solutions</u> to represent all of the solutions in term <u>of</u> linear combinations. $(c_1y_1 + \ldots + c_ky_k)$

<u>Def</u> Functions $f_1(x), f_2(x), \ldots f_n(x)$ are said to be <u>linearly dependent</u> if there exist constants, not all zero such that:
$c_1f_1(x) + c_2f_2(x) + \ldots + c_nf_n(x) \neq 0$ for all x.
(one of the functions can be written as a linear combination of the others)
Say $c_1 \neq 0$, $c_1f(x) = (-c_2)f_2(x) + \ldots + (-c_n)f_n(x)$
$f_1(x) = (-c_2/c_1)f_2(x) + (-c_3/c_1)f_3(x) + \ldots + (-c_n/c_1)f_n(x)$

Ex. Let $f_1(x) = 2\sin^2 x$, $f_2(x) = -3\cos^2 x$, $f_3(x) = 5$
Show that $f_1(x)$, $f_2(x)$, and $f_3(x)$ are linearly dependent.

Solution $\underbrace{(1/2)}_{c_1}\underbrace{(2\sin^2 x)}_{f_1} + \underbrace{(-1/3)}_{c_2}\underbrace{(-3\cos^2 x)}_{f_2} + \underbrace{(-1/5)}_{c_3}\underbrace{5}_{f_3}$

$1/2\, f_1(x) + (-1/3) f_2(x) + (-1/5) f_3(x) = 0$
$1/2\, f_1(x) = (1/3) f_2(x) + (1/5) f_3(x)$
$f_1(x) = (2/3) f_2(x) + (2/5) f_3(x)$

Thm. 3.3 Criterion for linearly independent solutions
Let $y_1, y_2, \cdots, y_n$ be solutions of a homo. eq'n over an interval I. Then $y_1, y_2, \ldots y_n$ are linearly independent.
$\Rightarrow W(y_1, y_2, \cdots y_n) \neq 0$ for all $x \in I$
where $W(y_1, y_2, \cdots y_n)$

$$= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \rightarrow \text{Wronskian of the functions}$$

(determinant)

Ex Show that $\{e^x, e^{2x}\}$ is linearly independent.
Solution By thm. 3.3 $W(e^x, e^{2x})$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}$$

$= e^x \cdot 2e^{2x} - e^x e^{2x}$
$= 2e^{3x} - e^{3x}$
$= e^{3x} \neq 0$

$\therefore W(e^x, e^{2x})$ is linearly independent.

$\{y_1, y_2, \cdots y_n\}$ is called a Fundemental set of solutions of $a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$ if it is linearly independent.

Thm. 3.4 (Existence of a Fundamental set)
There exists a Fundamental set of solutions.

Thm. 3.5 (General solution for homo.)
Set $y_1, y_2, \cdots, y_n$ be a fundamental set of solutions. The general solution
$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

Ex. Solve $y'' - 3y' + 2y = 0$ given that $y_1 = e^x$ and $y_2 = e^{2x}$ are two solutions.

Solution Since $W(e^x, e^{2x}) \neq 0$. As above, $\{e^x, e^{2x}\}$ is a fundamental set of solutions.
The general sol. $\boxed{y = c_1 e^x + c_2 e^{2x}}$

Ex. Given that $y_1 = e^x$, $y_2 = e^{2x}$, $y_3 = e^{4x}$ are solutions of $y''' - 5y'' + 2y' + 8y = 0$
Find the general solution.
Solution Verify that y_1, y_2, y_3 are linearly independent.

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^{-x} & e^{2x} & e^{4x} \\ -e^{-x} & 2e^{2x} & 4e^{4x} \\ e^{-x} & 4e^{2x} & 16e^{4x} \end{vmatrix}$$

$$= \begin{vmatrix} e^{-x} & e^{2x} & e^{4x} \\ 0 & 3e^{2x} & 5e^{4x} \\ e^{-x} & 4e^{2x} & 16e^{4x} \end{vmatrix}$$

$$= \begin{vmatrix} e^{-x} & e^{2x} & e^{4x} \\ 0 & 3e^{2x} & 5e^{4x} \\ 0 & 3e^{2x} & 15e^{4x} \end{vmatrix} \Rightarrow e^{-x} \begin{vmatrix} 3e^{2x} & 5e^{4x} \\ 3e^{2x} & 15e^{4x} \end{vmatrix}$$

$$= e^{-x}(45e^{6x} - 15e^{6x})$$

$$= e^{-x} \cdot 30(e^{6x}) = 30(e^{5x}) \neq 0$$

$\therefore$ they are linearly independent

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{4x}$$