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Applied Anal.

2.3 Linear Equations

Standard Form $\quad \frac{dy}{dx} + P(x)\, y = f(x)$

$$\frac{d}{dx}\left[e^{\int P(x)\,dx}\, y\right] = e^{\int P(x)\,dx} f(x)$$

$e^{\int P(x)\,dx}$ – an integrating factor

Ex: $\quad x\frac{dy}{dx} + (x-1)\,y = x^2 e^x$

Solution: First-order linear equation

(1) Standard Form: $\frac{dy}{dx} + \frac{x-1}{x}\cdot y = xe^x$

$P(x) : \frac{x-1}{x}, \quad f(x) = xe^x \qquad \left(e^{\alpha+\beta} = e^{\alpha}\cdot e^{\beta}\right)$

Remember!

(2) $d/dx\left[e^{\int P(x)\,dx}\cdot y\right] = e^{\int P(x)\,dx}\cdot f(x)$

An integrating factor: $e^{\int P(x)dx} = e^{\int \frac{x-1}{x}\,dx}$

$\Rightarrow e^{\int 1 - 1/x\, dx} \Rightarrow e^{x - \ln|x|}$

$\Rightarrow e^x \cdot e^{-\ln|x|}$

$= e^x \cdot (e^{\ln|x|})^{-1}$

$= e^x \cdot |x|^{-1}$

$\Rightarrow \frac{1}{x} e^x \quad (x > 0)$

$$\frac{d}{dx}\left[\left(\frac{1}{x}e^x\right)\cdot y\right] = \left(\frac{1}{x}\cdot e^x\right)\cdot(x \cdot e^x) = e^{2x}$$

$(1/x\, e^x)\, y = \int e^{2x}\,dx$

$1/x\, e^x y = 1/2\, e^{2x} + C$

$y = (1/2\, e^{2x} + C)\, x \cdot 1/e^x \Rightarrow \frac{1}{2} x e^x + Cxe^{-x}$

Ex. Solve $x\frac{dy}{dx} - 2y = x^3e^x$

Solution:

(1) Standard Form:

$$\frac{dy}{dx} - \frac{2}{x}y = x^2e^x$$

Note:

$$P(x) = \frac{-2}{x}$$

$$f(x) = x^2e^x$$

(2) $\frac{d}{dx}\left[e^{\int P(x)dx}\cdot y\right] \Rightarrow e^{\int P(x)dx}\cdot f(x)$

An integrating factor $e^{\int P(x)dx} = e^{\int -2/x\,dx}$

$= e^{-2\ln|x|} = (e^{\ln|x|})^{-2} \Rightarrow |x|^{-2} \Rightarrow \frac{1}{x^2}$

$$\frac{d}{dx}\left[\frac{1}{x^2}\cdot y\right] = \left(\frac{1}{x^2}\right)(x^2e^x) = e^x$$

$$\left(\frac{1}{x^2}y\right) = \int e^x dx = e^x + c$$

$$y = x^2e^x + cx^2$$

Ex. Solve $(x^2+2)\frac{dy}{dx} + xy = 0$

Solution: First order linear equation.

(1) Standard Form:

$$\frac{dy}{dx} + \frac{x}{(x^2+2)}\cdot y = 0$$

$$P(x): \frac{x}{(x^2+2)}$$

$$f(x): 0$$

(2) $\frac{d}{dx}\left[e^{\int P(x)dx}\cdot y\right] \Rightarrow e^{\int P(x)dx}\cdot f(x)$

$\mathscr{C}$

An integrating factor : $e^{\int \frac{x}{x^2+2}\,dx}$

$$\int \frac{x}{x^2+2}\,dx \Longrightarrow \frac{1}{2}\int \frac{1}{u}\,dx \Rightarrow \tfrac{1}{2}\ln|x^2+2| + c$$

$u = x^2+2$

$du = 2x\,dx$ $\Rightarrow e^{\frac{1}{2}\ln|x^2+2|}$

$dx = \tfrac{1}{2x}\,du$ $\Rightarrow (\cancel{e}^{\cancel{\ln}|x^2+2|})^{1/2}$

$\Rightarrow |x^2+2|^{1/2}$

$$d/dx \int \left[(x^2+2)^{1/2}\, y\right] = (x^2+2)^{1/2} \cdot 0 = 0$$

$$(x^2+2)^{1/2} = \int u\,dx = c$$

$$y = \frac{c}{(x^2+2)^{1/2}}$$

<u>Ex.</u> $(x^2+2)\,{}^{dy}/_{dx} + xy = 0$

— Separable!

$$(x^2+2)\frac{dy}{dx} = -xy$$

$$\int \frac{1}{y}\,dy = \int -\frac{x}{x^2+2}\,dx$$

$\ln|y| = -\tfrac{1}{2}\ln|x^2+2| + c_1$

$e^{\ln|y|} = e^{-\frac{1}{2}\ln|x^2+2| + c_1}$ $\rightarrow e^{c_1}\cdot e^{-\frac{1}{2}\ln|x^2+2|}$

$y = \pm e^{c_1}\left(e^{\ln(x^2+2)}\right)^{-1/2}$

$y = \pm c\,(x^2+2)^{-1/2} \Rightarrow \dfrac{c}{(x^2+2)^{1/2}}$

Ex. Solve the IVP : $\begin{cases} dy/dx + 2xy = x \\ y(0) = -3 \end{cases}$

Solution : $dy/dx + 2xy = x$ is a linear equation!
(it's in its standard form)

$P(x) = 2x, \quad f(x) = x$

$$\frac{d}{dx}\left[e^{\int P(x)\,dx} \cdot y\right] = e^{\int P(x)\,dx} \cdot f(x)$$

An integrating factor $e^{\int P(x)\,dx} = e^{\int 2x\,dx}$
$\Rightarrow e^{x^2}$

$$\left[\frac{d}{dx}\right]\left[e^{x^2} \cdot y\right] = e^{x^2} \cdot x$$

$u = x^2$
$du = 2x\,dx$
$\tfrac{1}{2}du = x\,dx$

$$e^{x^2}y = \int e^{x^2} \cdot x\,dx = \int e^u \cdot (\tfrac{1}{2}du)$$
$$= \tfrac{1}{2}e^u + C = \tfrac{1}{2}e^{x^2} + C$$
$$y = (\tfrac{1}{2}e^{x^2} + C)\,\tfrac{1}{e^{x^2}} = \tfrac{1}{2} + C \cdot e^{-x^2}$$

IVP : $y(0) = -3, \quad \tfrac{1}{2} + C \cdot e^{-0^2} = -3$

$\tfrac{1}{2} + C = -3 \qquad -C = \tfrac{1}{2} + 3 = -7/2$

$y = \tfrac{1}{2} + \tfrac{7}{2}e^{-x^2}$ is the solution of the IVP.

Ex. Solve the IVP $x\frac{dy}{dx} + y = 2x$

Solution : $x\frac{dy}{dx} + y = 2x$ is a linear equation

(1) Standard Form :
$$\frac{dy}{dx} = \frac{y}{x} = 2$$

(2) $\frac{d}{dx}\left[e^{\int P(x)dx}\, y\right] = e^{\int P(x)dx} \cdot f(x)$

Note:
$P(x) = 1/x$
$f(x) = 2$

An integrating Factor:

$\Rightarrow e^{\int P(x)dx} = e^{\int 1/x \, dx}$

$\Rightarrow e^{\ln|x|} = |x| = x \quad (x > 0)$

$\frac{d}{dx}\left[\int (x \cdot y) \right] = (x)(2) = 2x$

$\hookrightarrow (x \cdot y) = \int 2x \, dx = x^2 + C$

$y = x + \frac{C}{x} \rightarrow$ 1-parameter Family of Solutions

IVP: $y(1) = 0$ / $1 + C = 0$, $C = -1$

$1 + C/1 = 0$

$y = x + -1/x$ is the solution

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2.4 Exact Equations

(1) Separable $dy/dx = f(x)\,g(y)$

(2) Linear Equation

$dy/dx + p(x)y = f(x)$

Differentials !

(1) Single-variable Function $(f(x))$

$d\,f(x) = f'(x)\,dx$ $\qquad dy/dx = f'(x)$

$\frac{d\,f(x)}{dx} = f'(x)$ $\qquad dy = f'(x)\,dx$

$df(x) = f'(x)\,dx$

(2) Double Variable Function $f(x,y)$

$d\,f(x,y) = \left(\frac{df}{dx}\right)dx + \left(\frac{df}{dy}\right)dy$

Example: if $f(x,y) = 2xy^2 + e^x y + (\sin x)\,y$

$\partial f/\partial x = 2y^2 + ye^x + y\cos(x)$

$\partial f/\partial y = 2x2y + e^x + \sin(x)$

$df(x,y) = \left(\frac{df}{dx}\right)dx + \left(\frac{df}{dy}\right)dy$

$df = (2y^2 + ye^x + y\cos x)\,dx + (4xy + e^x + \sin x)\,dx$

* Property if $df(x,y) = 0$, then $f(x,y) = c$ (constant)

Ex. Solve $\frac{dy}{dx} = \frac{-2y}{2x+4y^3}$

Solution: $(2x + 4y^3)\,dy = -2y\,dx$

$2y\,dx + (2x + 4y^3)\,dy = 0$

$d(2xy + y^4) = 0$

$\partial/\partial x\,(2xy + y^4) = 2y$; $\partial/\partial y\,(2xy + y^4) = 2x + 4y^3$

$2xy + y^4 = c$ is the solution

Generally, to solve

$$\left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy = 0$$

$$d f(x, y) = 0$$

The solution is $f(x, y) = C$

Definition $M(x,y)\, dx + N(x,y)\, dy = 0$

is said to be an exact DE if

$M(x,y)\,dx + N(x,y)\,dy = d f(x,y)$

For some $f(x,y)$, i.e. there exists

$f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M(x,y), \quad \frac{\partial f}{\partial y} = N(x,y)$$

Then the equation becomes

$$\left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy = 0, \quad \text{i.e}$$

$$d f(x,y) = 0$$

$f(x,y) = C$ is in family of solutions with parameter C.

How to find $f(x,y)$ if it is exact

$$\begin{cases} \frac{\partial f}{\partial x} = M(x,y) \\ \frac{\partial f}{\partial y} = N(x,y) \end{cases}$$ — Solve it for $f(x,y)$!

Thm 2.1 Criterion

$M(x,y)\,dx + N(x,y)\,dy = 0$ is exact

$$\Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Ex. Solve $(1+2x+y^2)\,dx + (2x+3)\,dy = 0$

Solution: $M(x,y) = 1+2x+y^2$; $\frac{dM}{dy} = 2y$

$N(x,y) = 2xy+3$; $\frac{dN}{dx} = 2y$

$\therefore$ it's exact, ie $\exists\, f(x,y)$ s.t.

$$\begin{cases} \frac{df}{dx} = 1+2x+y^2 \quad - (1) \quad (*) \\ \frac{df}{dy} = 2xy+3 \quad - (2) \end{cases}$$

From (1), $f(x,y) = \int 1+2x+y^2\,dx$

$= x + x^2 + y^2x + c(y)$

$\frac{df}{dy} = 2yx + c'(y)$ From (2)

got

$2xy+3 = 2xy + c'(y)$

$c'(y) = 3$ $\quad c(y) = \int 3\,dy = 3y$

$f(x,y) = x + x^2 + xy^2 + 3y$

The solution $x + x^2 + xy^2 + 3y = C$

Solution Exact?

$M(x,y) = 2x^2 + y$. $\frac{dM}{dy} = 2y$

$N(x,y) = x^2 + 2y$ $\quad \frac{\partial N}{dx} = 2x$

It is not exact since $\frac{\partial M}{dy} \neq \frac{dN}{dx}$

$$\begin{cases} \dfrac{df}{dx} = 2x^2 + y^2 \\ \dfrac{df}{dy} = x^2 + 2y \end{cases}$$

has no solution for $f(x, y)$

Ex. Solve : $(y^2 \cos x - 3x^2 y - 2x)dx + (2y \sin x - x^3 + \ln y)dy = 0$

Solution Exact ?

$M = y^2 \cos x - 3x^2 y - 2x \qquad \dfrac{dM}{dy} = 2y \cos x - 3x^2$

$N = 2y \sin x - x^3 + \ln y \qquad dN/dx = 2y \cos x - 3x^2$

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2.4 Exact Equations

$M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if there is a function $f(x,y)$ such that

$$\begin{cases} \delta f/\delta x = M(x,y) \\ \delta f/\delta y = N(x,y) \end{cases}$$

Then the DE is $(\delta f/\delta x)\,dx + (\delta f/\delta y)\,dy = 0$

i.e. the DE is $d f(x,y) = 0$

$f(x,y) = c$ is the solution

Criterion: it is exact $\Longleftrightarrow \dfrac{\delta M}{dy} = \dfrac{\delta N}{dx}$

Ex. Solve IVP

$[(y^2\cos x - 3x^2y) - 2x]\,dx + (2y\sin x - x^2 + \ln x)\,dy = 0$

Solution $M(x,y) = y^2\cos x - 3x^2y - 2x$

$\dfrac{\partial M}{dy} = 2y\cos x - 3x^2$ $\therefore$ it is exact

There is $f(x,y)$ such that

$$\begin{cases} \delta f/\delta x = y^2\cos x - 3x^2y - 2x & -\ (1) \\ \delta f/\delta y = 2y\sin x - x^3 + \ln y & -\ (2) \end{cases}$$

From (1), $f(x,y) = \int y^2\cos x - 3x^2y - 2x\,dx$

$= y^2\sin x - yx^3 - x^2 + c(y)$

$= y^2\int\cos x\,dx - y\int 3x^2\,dx - \int 2x\,dx$

For this $f(x,y)$, $\dfrac{\delta f}{dy} = 2y\sin x - x^3 + c'(y)$

From (2), $\cancel{2y}\sin x - \cancel{x^3} + c'(y) = \cancel{2y}\sin x - \cancel{x^3} + \ln(y)$

$c'(y) = \ln(y)$

$\Rightarrow c(y) = \int \ln(y)\,dx$

$c(y) = y\ln y - y$

$\int u\,dv = uv - \int v\,du$

$\int \ln y\,dy$

$u = \ln y$; $u' = \frac{1}{y}dy$

$v = 1$; $v' = dy$

$\therefore f(x,y) = y^2\sin x - yx^3 - x^2 + \ln y \cdot y - y$

$y^2\sin x - yx^3 - x^2 + y\ln y - y = C$

is the implicit solution of the ~~ivp~~ DE

IVP: $y(0) = e$, $e^2\sin(0) - e\cdot 0^3 - 0^2 \ldots$

$\ldots e\ln e - e$

$e - e = c \quad \therefore \quad C = 0$

$y^2\sin x - yx^3 - x^2 + y\ln y - y = 0$

is the solution of the IVP

Although $M dx + N dy = 0$ is not exact

$(\mu(x) M)\,dx + (\mu(x) N)\,dy = 0$

is exact. $\mu(x)$ - an integrating factor.

Criterion : $\frac{\partial}{dy}(\mu(x) M) = \frac{\partial}{dx}(\mu(x) N)$

$\mu(x)\,\partial/dy\, M = N\,d\mu/dx + M\frac{\partial N}{dx}$ [Solve it for μ x-variable]

$N\frac{d\mu}{dx} = -\mu\frac{\partial N}{\partial x} + \mu\frac{dM}{dy}$

$1/\mu \cdot du = 1/N\left(-\frac{dN}{dx} + \frac{dM}{dy}\right)dx$

(Assume: $1/N\left(\frac{\partial N}{dx} - \frac{dM}{dy}\right)$ (is in x only)

$\ln|\mu| = \int 1/N\,(dN/dx - dM/dy)\,dx$

$u = e^{\int 1/N\left(\frac{dN}{dx} + \frac{dM}{dy}\right)dx}$

$$\boxed{\mu(x) = e^{\int 1/N\,(dM/dy - dN/dx)\,dx}}$$

Ex. Solve $(2y^2 + 3x)\,dx + 2xy\,dy = 0$

Solution $M(x,y) = 2y^2 + 3x$, $\dfrac{dM}{dy} = 4y$

$N(x,y) = 2xy$, $\dfrac{dN}{dx} = 2y$

Since $\dfrac{dM}{dy} \neq \dfrac{dN}{dx}$, it's not exact

An integrating Factor

$\mu(x) = e^{\int \frac{1}{N}\left(\frac{dM}{dy} - \frac{dN}{dx}\right) dx}$

$\frac{1}{N}\left(\frac{dM}{dy} - \frac{dN}{dx}\right) = \frac{1}{2xy}(4y - 2y) = \frac{1}{x}$

$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = |x| = x \quad (x > 0)$

$x\left((2y^2 + 3x)dx + 2xy\, dy\right) = 0$

$(2xy^2 + 3x^2)dx + 2x^2 y\, dy = 0$

$\left[\frac{d}{dy}(2xy^2 + 3x^2) = 4xy, \quad \frac{d}{dx}(2x^2y) = 4xy \right]$

$$\begin{cases} \frac{df}{dx} = 2xy^2 + 3x^2 & -(1) \\ \frac{df}{dy} = 2x^2 y & -(2) \end{cases}$$

From (1), $f(x,y) = \int 2xy^2 + 3x^2\, dx$

$= y^2 \int 2x\, dx + \int 3x^2\, dx$

(this is what he wrote, what is this?) $= y^2 x^2 + x^3 + C(y)$

For this $f(x,y)$; $\dfrac{df}{dy} = 2x^2 y + C'(y)$

From (2) $\frac{df}{dy} = 2x^2 y$

$2x^2y + C'(y) = 2x^2 y$; $C'(y) = 0$

$C(y) = C.$

$f(x,y) = x^2y^2 + x^3$

$x^2y^2 + x^3 = C$

2.7 Linear Models

(1) Growth and decay : $x(t)$ - amount

dx/dt - rate of growth

* $dx/dt \rightarrow$ is proportional to $x(t)$!

$dx/dt = kx$, $x(t_0) = x_0$

The rate of bacteria growth is proportional to the number of bacteria $N(t)$ present. If we know $N(0) = 1000$, and $N(1) = 5/4\, N(0)$ Find the time t at which the number of bacteria is doubled.

Solution: (1) Solve $\frac{dN}{dt} = kN$

$\frac{dN}{dt} - kN = 0$ - linear equation

$d/dt\left(e^{\int P(t)dt} \cdot N\right) = e^{\int P(t)dt} \cdot f(x)$

$P(t) = -k$, $f(t) = 0$