Sept. 15/17

Applied Anal

Ex. Solve the IVP

$y/x \; dy/dx = (1+x^2)^{-1/2}(1+y^2), \quad y(0) = 0$

Solution $\int \frac{y\,dy}{(1+y^2)} = \int (1+x^2)^{-1/2}\, x\,dx$

$\Rightarrow \int \frac{y}{1+y^2}\,dy \xrightarrow[u=1+y^2]{} \int \frac{1}{u}\cdot(1/2\,du) \Rightarrow \frac{1}{2}\ln|u|$

$\Rightarrow \frac{1}{2}\ln|1+y^2| + c$

$du = 2y\,dy$

$1/2\,du = y\,dy$

$\Rightarrow (1+x^2)^{-1/2}\,x\,dx \xrightarrow[u=1+x^2]{} \int u^{-1/2}\cdot(1/2\,du)$

$du = 2x\,dx \qquad \Rightarrow \int u^n\,du = \frac{u^n}{n+1}$

$1/2\,du = x\,dx$

$\Rightarrow \frac{1}{2}\cdot\frac{u^{-1/2+1}}{-1/2+1} + C_2$

$= u^{\frac{1}{2}} + C_2 = \sqrt{1+x^2} + C_2$

$= 1/2\ln(1+y^2) = \sqrt{1+x^2} + C$

– Family of solutions

IVP: $y(0) = 0$, if $x = 0$, then $y = 0$

$\frac{1}{2}\ln(1+0^2) = \sqrt{1+0^2} + C$

$\frac{1}{2}\times 0 = 1 + C \quad \therefore \quad C = -1$

So the solution of the IVP is

$1/2\ln(1+y^2) = \sqrt{1+x^2} - 1.$

2.3 Linear Equations

$a_1(x)y' + a_2(x)y = g(x)$ – general form

$dy/dx + \left(\frac{a_2(x)}{a_1(x)}\right)\cdot y = \frac{g(x)}{a_1(x)}$

Standard Form:

$dy/dx + P(x)y = f(x)$

Formula: $\frac{d}{dx}\left[e^{\int P(x)dx} \cdot y\right] = e^{\int P(x)dx} \cdot f(x)$

Where is this Formula From?

If $f(x) \neq 0$, $dy/dx + P(x)y = f(x)$ is
non-homogeneous

$dy/dx + P(x)y = 0$ is homogeneous

* Let y_p be a particular of $dy/dx + P(x)y = f(x)$
Let y_c be a family of solutions of $dy/dx + P(x)y = 0$
Then the general solutions of $dy/dx + P(x)y = f(x)$
is $y = y_c + y_p$

(1) Find y_c: $dy/dx + P(x)y = 0$

$dy/dx = -P(x)y$, $\int 1/y \, dy = \int -P(x)dx + C_1$

$\ln|y| = -\int P(x)dx$, $e^{\ln|y|} = e^{-\int P(x)dx + C_1}$

$|y| = e^{-\int P(x)dx} \cdot e^{C_1}$; $y = \pm e^{C_1} e^{-\int P(x)dx}$ ($\pm e^{C_1}$ → constant)

$y = C e^{-\int P(x)dx}$

$y_c = C \cdot e^{-\int P(x)dx}$

(2) To Find y_p: $dy/dx + P(x)y = f(x)$

Denote $y_1 = e^{-\int P(x)dx}$ let $y_p = U(x)y_1$
be a particular solution; variation + parameter

Find a function $U(x)$ such that

$d/dx(uy_1) + P(x)(uy_1) = f(x)$

$uy_1' + u'y_1 + P(x)(uy_1) = f(x)$

$u(y_1' + P(x)y_1) + u'y_1 = f(x)$

Since y_1 is a solution of $dy/dx + P(x)y = 0$

$u'y_1 = f(x)$, $u' = 1/y_1 \cdot f(x)$

$u = \int 1/y_1 \cdot f(x)dx$; $y_p = uy_1 = \left(\int \frac{1}{y_1} f(x)\right) y_1$

Note $y_1 = e^{-\int P(x)dx}$, $1/y_1 = e^{\int P(x)dx}$

$y_p = y_1 \int e^{\int P(x)dx} \cdot f(x)dx$

The general solution is

$$y = y_c + y_p$$
$$= c e^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{-\int P(x)dx} \cdot f(x)dx$$

$\times\ e^{\int P(x)dx}$

$$e^{\int P(x)dx} \cdot y = C + \int \left[e^{\int P(x)dx} \cdot f(x) \right] dx$$

$$d/dx \left[e^{\int P(x)dx} \cdot y \right] = d/dx \left[C + \int e^{\int P(x)dx} f(x)dx \right]$$

$$d/dx \left[e^{\int P(x)dx} \cdot y \right] = e^{\int P(x)dx} f(x)$$

Ex. Solve $x^3\, dy/dx - 2x^2 y = 1$

Solution: First Order Linear Equation!

Standard Form: $\frac{dy}{dx} - \frac{2x^2}{x^3} \cdot y = \frac{1}{x^3}$

$f(x) = 1/x^3$, $P(x) = \frac{-2x^2}{x^3}$ Carry "-"!

$$= -\frac{2}{x}$$

$e^{\int P(x)dx}$ - integrating Factor

$e^{\int -2/x\, dx} = e^{-\int 2/x\, dx} = e^{-2\ln|x|}$ (take one only)

(Simplify that!) $\longrightarrow$ $e^{\alpha B} = (e^B)^\alpha$

$= (e^{\ln|x|})^{-2} \to |x|^{-2}$

$\to 1/|x|^2 \to 1/x^2$

$$d/dx \left(e^{\int P(x)dx} \cdot y \right) = e^{\int P(x)dx} \cdot f(x)$$

$d/dx(1/x^2 \cdot y) = (1/x^2)(1/x^3) = x^{-5}$

$(1/x^3 \cdot y) = \int x^{-5} dx = \frac{x^{-4}}{-4} + C$

$y = -1/4\, x^{-1} + Cx^3$ ✓