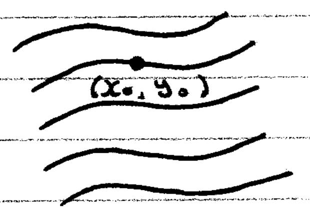
1.2 Initial Value Problems (IVP)

First-order IVP $dy/dx = f(x,y)$

Subject to $y(x_0) = y_0$



Example: Solve the IVP $\begin{cases} dy/dx = 2x \\ y(0) = 1 \end{cases}$

Solution: $y = \int 2x\,dx \Rightarrow x^2 + C$

$y = x^2 + C \Rightarrow 0^2 + C = 1 \ ; \ C = 1$

if $x = 0,\ y = 1$

$\therefore\ y = x^2 + 1$ is the solution of the IVP

Second order IVP $\dfrac{d^2y}{dx^2} = f(x, y, y^2)$

Subject to $y(x_0) = y_0$, $y'(x_0) = y_1$

(position) (velocity)

Example: Solve IVP $\begin{cases} y'' + 16y = 0 \\ y(\pi/2) = -2,\ y'(\pi/2) = 1 \end{cases}$

Given that the solution of $y'' + 16y = 0$

is $y = C_1 \cos 4x + C_2 \sin 4x$

Solution: $y(\pi/2) = -2$: $C_1 \cos 4 \times \pi/2 + C_2 \sin 4x \times \pi/2 = -2$

$C_1 \cos 2\pi + C_2 \sin 2\pi = -2$

$C_1 + 0 = -2$ $\boxed{C_1 = -2}$

$y'(\pi/2) = 1$: $\quad y' = -4C_1 \sin 4x + 4C_2 \cos 4x$

$$-4C_1 \sin 4x \times \pi/2 + 4C_2 \cos 4x \times \pi/2 = 1$$

$$0 + 4C_2 = 1, \quad C_2 = 1/4$$

$y = -2\cos 4x + 1/4 \sin 4x$ is the solution of the IVP

3rd IVP :
$$\begin{cases} \dfrac{d^3y}{dx^3} = f(x, y, y', y'') \\ y(x_0) = y_0, \quad y'(x_0) = y_1, \quad y''(x_0) = y_2 \end{cases}$$

Does IVP have a solution?

Sometimes there is no solution

e.g. $\begin{cases} (y')^2 + 1 = 0 \\ y(0) = 0 \end{cases}$

Sometimes the solution is not unique

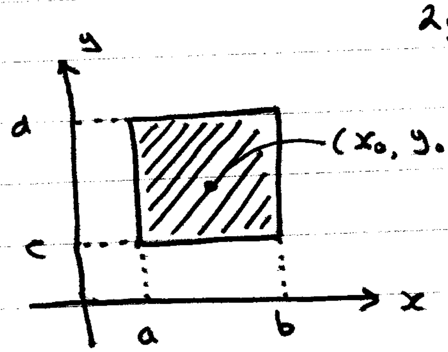
e.g. $\begin{cases} dy/dx = xy^{1/2} \\ y(0) = 0 \end{cases}$ has two solutions

$y = 0$ and $y = x^4/16$

Thm 1.1 Let $dy/dx = f(x, y)$, $y(x_0) = y_0$

and $R = \{(x, y) : a \le x \le b, \ 0 \le y \le b\}$

If $f(x,y)$ and $\dfrac{\partial F}{\partial y}$ are continuous on R, then the IVP has a unique solution.



Ex. $f(x,y) = x^2 + 2xy - \sin(x+y+1)$

$$\frac{\partial f}{\partial y} = \emptyset + 2x - \cos(x+y+1)$$

Ex. $f(x,y) = xe^{xy} + y$

$$\frac{\partial f}{\partial y} = x\,(e^{xy})' + y' = xe^{xy}\cdot x + 1$$

1.3 DES as mathematical models

(1) Growth + Decay :

$x(t)$ - the amount of t.

the rate of growth is proportional to the amount of any time t

$$\frac{dx}{dt} \sim x(t) \qquad \frac{\left[\frac{dx}{dt}\right]}{x(t)} = k$$

$$\boxed{\frac{dx}{dt} = Kx} \qquad x(t_0) = x_0$$

(2) Spread of disease

$x(t)$ - the # of people who have the disease

$y(t)$ - the # of people who do not have the disease

* the rate at which the disease spreads is proportional to the product of x and y

$$\boxed{dy/dx = Kxy}$$

→ If a community has n people and one infected person is into the community

$x(0) = 1$; $x + y = n+1$; $y = n+1-x$

$$\boxed{dx/dt = kx(n+1-x), \quad x(0) = 1}$$

3) Newton's Law of Cooling

$T(t)$ - temperature at t

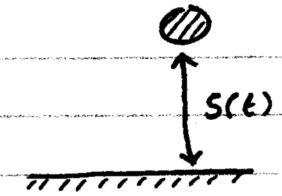
T_m - the temperature of the surrounding medium

The rate at which a body cools is proportional to $T - T_m$

$$\boxed{\frac{dT}{dt} = k(T - T_m)}$$

Newton's Second Law:

$$\frac{d^2s}{dt^2} = -g$$



initial value condition

$s(0) = s_0$, $s'(0) = s_1$

Chapter 2 - First-Order DE's

2.2 Separable Variables

Ex. Solve $dy/dx = f(x)$

Solution y - antiderivatives of $f(x)$

$y = \int f(x)\,dx \longrightarrow$ Family of Functions

Ex. Solve $dy/dx = \sin x + 3x^2 + e^{-x}$

Solution $y = \int \sin x + 3x^2 + e^{-x}\,dx$

$= -\cos x + x^3 - e^{-x} + C$

Ex. Solve $dy/dx = xy$

Solution $\cancel{y = \int xy\,dx\ ??}$

Definition: Seperable DE

$dy/dx = f(x)\,g(x)$

Example: $dy/dx = \sin(xy) + y$ is not separable

Ex. $dy/dx = xy$ is Seperable

How to solve a separable equation?

$dy/dx = f(x)\,g(x) \longleftarrow 1/g(x)\ dx$

$1/g(x)\ dy = f(x)\,dx$

$\int \frac{1}{g(y)} = \int f(x)\,dx$

Ex. Solve $dy/dx = xy$

Solution $\frac{1}{y}\, dy = x\, dx$, $\frac{1}{x}$, dx

$$\int \frac{1}{y}\, dy = \int x\, dx$$

$$\ln|y| + C_1 = \frac{x^2}{2} + C_2$$

$$\ln|y| + C_1 = \frac{1}{2}x^2 + C_2$$

$$\ln|y| = \frac{1}{2}x^2 + C_2 - C_1$$

$$\ln|y| = \frac{1}{2}x^2 + C$$

↪ a Family of Solutions with one parameter C

Any solution from the First Family,

Say $C_1 = 2$, $C_2 = 10$

$$\ln|y| = \frac{1}{2}x^2 + 10 - 2$$

$$\ln|y| = \frac{1}{2}x^2 + 8$$

$\{C_2 - C_1 : C_1, C_2$ are numbers $\}$

$= \{C : C$ is a number $\} = \mathbb{R}$

$\cdot \{e^c : c$ is a number $\}$ = the set of all nonzero numbers

Ex. Solve $dy/dx = \frac{y}{2+x}$ and Find the explicit solutions.

Solution $\frac{1}{y}\, dy = \frac{1}{2+x}\, dx$

$$\int \frac{1}{y}\, dy = \frac{1}{2+x}\, dx$$

$$\ln|y| = \ln|2+x| + c$$

$$e^{\ln|y|} = e^{\ln|2+x| + c}$$

$$|y| = e^c \cdot |2+x|$$

$$y = \pm e^c (2+x) , \qquad y = C(2+x)$$

↪ explicit Solution

Ex. Solve $(1+y^2)e^x \frac{dy}{dx} = xy$

Solution: $\int (1+y^2)\, ^1/_y \, dy = \int x \, ^1/_{e^x} \, dx$

$\int {}^1/_y + y \, dy = \int x e^{-x} dx$

$\ln|y| + {}^1/_2 y^2 = -xe^{-x} + e^{-x} + c$

$\boxed{\int u\, dv = uv - \int v\, du}$

$\int x e^{-x} dx = (x)(-e^{-x}) - \int (-e^{-x}) dx$

$= -xe^{-x} + \int e^{-x} dx = -xe^{-x} + e^{-x} + C$

* a constant solution may be lost if a divisor is zero when separating variables

Ex. Solve $dy/dx = y^2 - 1$

Solution divide both sides of the DE by y^2-1 — divisor

(if $y^2 - 1 = \emptyset$, i.e. $y = \pm 1$, then the divisor is zero)

then $y \pm 1$ may be the lost solutions.

$\frac{1}{y^2-1} dy = dx$

$\int \frac{1}{y^2-1} dy = \int$

$\int \frac{1}{(y+1)(y-1)} dy = x + c$

$\rightarrow \int \frac{1}{(y+1)(y-1)} = \frac{A}{(y+1)} + \frac{B}{(y-1)}$ partial fractions

$1 = A(y-1) + B(y+1)$

1 $y=1 : 1 = \emptyset + B(1+1)$ $B = {}^1/_2$

$y=-1 : \rightarrow$ $A = -{}^1/_2$

$$\int \frac{-1/2}{y+1} + \frac{1/2}{y-1}\, dy = x + C$$

$$-\tfrac{1}{2} \ln|y+1| + \tfrac{1}{2} \ln|y-1| = x + C$$

↳ implicit solution

$y = 1$ and $y = -1$ are two solutions not in this family (lost)