Nov. 27/17

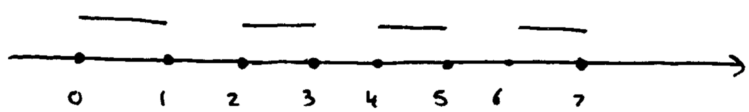
Applied Anal.

Let $f(t)$ be a particular function with period $T$,

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-sT} f(t)\,dt$$

$$\mathcal{L}\{E(t)\} = \frac{1}{s(1+e^{-sT})}$$

where $\{E(t)\}$ is the square wave function




<u>Example</u> (A Periodic Impressed Voltage)

The DE for the current $i(t)$ in a single-loop LR-series circuit

$$L\frac{di}{dt} + Ri = E(t)$$

Determine the current $i(t)$ when $i(0) = 0$ and $E(t)$ is the square wave function

<u>Solution</u> $L\mathcal{L}\{\frac{di}{dt}\} + R\{i\} = \mathcal{L}\{E(t)\}$

$$L(s\,I(s) - i(0)) + R\,I(s) = \frac{1}{s(1+e^{-s})}$$

$$(Ls + R)\,I(s) = \frac{1}{s(1+e^{-s})}$$

$$I(s) = \frac{1}{(Ls+R)(1+e^{-s})s}$$

$$i(t) = \mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(Ls+R)} \cdot \frac{1}{1+e^{-s}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{L} \cdot \frac{1}{s(s+R/L)} \cdot \frac{1}{1+e^{-s}}\right\}$$

$$= \frac{1}{L}\mathcal{L}^{-1}\underbrace{\left\{\frac{1}{s(s+R/L)} \cdot \frac{1}{1+e^{-s}}\right\}}_{\text{PARTIAL FRACTION}}$$

$$\frac{1}{(s)(s+R/L)} = \frac{A}{s} + \frac{B}{(s+R/L)} \quad (\text{omit}) \Rightarrow \frac{(L/R)}{s} - \frac{(L/R)}{s+R/L}$$

$$i(t) = \frac{1}{L}\mathcal{L}^{-1}\left\{\left(\frac{L}{R}\cdot\frac{1}{s} - \frac{L}{R}\cdot\frac{1}{s+R/L}\right)\frac{1}{1+e^{-s}}\right\}$$

$$\Rightarrow \frac{1}{L}\cdot\frac{L}{R}\mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{1}{s+R/L}\right)\frac{1}{1+e^{-s}}\right\}$$

$$\Rightarrow \frac{1}{R}\mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{1}{s+R/L}\right)\left(\frac{1}{1+e^{-s}}\right)\right\}$$ ("bring $e^{-s}$ up")

geometric series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 \dots$$

$x = e^{-s}$ , $|e^{-s}| < 1$

$$\frac{1}{1+e^{-s}} = 1 - e^{-s} + e^{-2s} - e^{-3s} + e^{-4s} \dots$$

$$i(t) = \frac{1}{R} \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{1}{s}e^{-s} + \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-3s} + \dots\right) \dots\right.$$

$$\left. \dots - \left(\frac{1}{s+R/L} - \frac{1}{s+R/L}e^{-s} + \frac{1}{s+R/L}e^{-2s} \dots\right)\right\}$$

$$\Rightarrow \frac{1}{R}\left[\left(1 - \mathcal{U}(t-1) + \mathcal{U}(t-2) - \mathcal{U}(t-3) \cdots\right] - \left(e^{-R/L\,t} - e^{-R/L\,t-1}\mathcal{U}(t-1) \dots\right.\right.$$

$$\left. \dots + e^{-R/L\,t-2}\mathcal{U}(t-2) - e^{-R/L\,t-3}\mathcal{U}(t-3) \cdots\right]$$

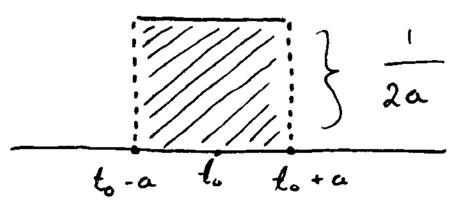
$$i(t) = \frac{1}{R}\left(1 - e^{-R/L\,t}\right) + \frac{1}{R}\sum_{n=1}^{\infty}(-1)^n\left(1 - e^{-R/L\,(t-n)}\right)\mathcal{U}(t-n)$$

## 4.5 The Dirac Delta Function

Unit impulse $a > 0$ , $t_0 > 0$

$$\delta_a(t-t_0) = \begin{cases} 0 , & 0 \le t \le t_0 - a \\ 1/2a , & t_0 - a \le t \le t_0 + a \\ 0 , & t \ge t_0 + a \end{cases}$$

$$\int_{-\infty}^{\infty} \delta_a(t-t_0)\,dt = 1$$




The Dirac Delta Function

$$\delta(t-t_0) = \lim_{a\to 0} \delta_a(t-t_0)$$

It is characterized by two properties

a) $\delta(t-t_0) = \begin{cases} \infty , & \text{if } t = t_0 \\ 0 , & \text{if } t \ne 0 \end{cases}$

b) $\int_0^{\infty} \delta(t-t_0)\,dt = 1$

Define the Laplace transform For $\delta(t-t_0)$

$$\mathcal{L}\{\delta(t-t_0)\} = \lim_{a\to 0} \mathcal{L}\{\delta(t-t_0)\}$$

Thm. 4.5.1 For $t_0 > 0$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$$

Inverse $\mathcal{L}^{-1}\{e^{-st_0}\} = \delta(t-t_0)$

Ex Solve the IVPs :

(a) $y'' + y = 4\delta(t-2\pi)$, $y(0) = 1$, $y'(0) = 0$

(b) $y'' + y = 4\delta(t-2\pi)$, $y(0) = 0$, $y'(0) = 0$

Solution (a) $\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 4\,\mathcal{L}\{\delta(t-2\pi)\}$

$$(s^2 Y(s) - s\,y(0) - y'(0)) + Y(s) = 4e^{-s\cdot 2\pi}$$

$$s^2 Y(s) - s + Y(s) = 4e^{-2\pi s}$$

$$(s^2+1)Y(s) = s + 4e^{-2\pi s}$$

$$Y(s) = \frac{s}{s^2+1} + 4e^{-2\pi s}\left(\frac{1}{s^2+1}\right)$$

$$y(t) = \mathcal{L}\left\{\frac{s}{s^2+1}\right\} + 4\,\mathcal{L}^{-1}\left\{\left(e^{-2\pi s}\right)\left(\frac{1}{s^2+1}\right)\right\}$$

$$= \cos t + 4\,f(t-2\pi)\,\mathcal{U}(t-2\pi)$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$$y = \cos t + 4\sin(t-2\pi)\,\mathcal{U}(t-2\pi)$$

$$y = \cos t + 4\sin(t)\,\mathcal{U}(t-2\pi)$$

b) $(s^2 Y(s) - s\,y(0) - y'(0)) + \mathcal{L}\{y\} = 4e^{-2\pi s}$

$$(s^2+1)Y(s) = 4e^{-2\pi s}$$

$$Y(s) = \frac{4e^{-2\pi s}}{s^2+1}, \quad y = \mathcal{L}^{-1}\left\{\frac{4e^{-2\pi s}}{s^2+1}\right\}$$

$$= 4\sin t\,\mathcal{U}(t-2\pi)$$

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APPLIED ANAL.

Chapter 1: Basic Concepts, Terminoloy

Chapter 2: First Order Eq'n

(1) Seperable equations

$$\frac{dy}{dx} = g(x)h(y)$$

$$\int \frac{1}{h(y)}\, dy = \int g(x)dx$$

(2) Linear Equations

$$a_1(x)\, dy/dx + a_0(x)\, y = g(x)$$

$\Downarrow$ Standard Form

$$\frac{dy}{dx} + P(x)y = f(x)$$

$$d/dx\left[e^{\int P(x)dx}\cdot y\right] = e^{\int P(x)dx}\cdot f(x)$$

(3) Exact Equations

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact $\Leftrightarrow \dfrac{\delta M}{\delta y} = \dfrac{\delta N}{\delta x}$

$$\begin{cases} \dfrac{\delta f}{\delta x} = M & \text{— (1)} \\ \dfrac{df}{\delta y} = N & \text{— (2)} \end{cases} \qquad \begin{array}{l} \delta f(x,y) = 0 \\ \\ f(x,y) = c \end{array}$$

2.7 - Linear Models

(1) Growth and Decay : $x(t)$ - amount of something

$\frac{dx}{dt}$ is proportional to the amount,

i.e. $dx/dt = kx$, $x(t_0) = x_0$

half-life : the time $t$ at which $x(t) = \frac{1}{2}x_0$

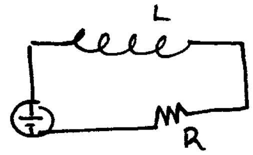
(2) Newton's Law of cooling :

$T(t)$ - temp of object (of a body)

$$\frac{dT}{dt} = k(T - T_m)$$

$\hookrightarrow$ the temperature of the medium

(3) Series Circuits :




$$L \, di/dt + Ri = E(t)$$

$$\left[ \frac{dy}{dx} + P(x)y = f(x) \right]$$

# Chapter 3 - higher order DE

3.1 - Linear equations : basic theory

$$a_3(x)y''' + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

(1) Solve the associated homo. eq'n. (because = zero)

$$a_3(x)y''' + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

i.e. Find linearly indep. solutions

$y_1, y_2, y_3$ of the general solution

$$y = c_1y_1 + c_2y_2 + c_3y_3$$

(2) Find a particular solution $y_p$ for the non-homo.

(3) The general solution for the non-homo

$$y = c_1y_1 + c_2y_2 + c_3y_3 + y_p$$

3.2 - reduction of order

$$a_2(x)y'' + a(x)y' + a_0(x)y = 0$$

If we got a solution, $y_1 \neq 0$

How to find a second one

$y_2 = y_1 u(x)$ for some function

.... (long calculation)

Standard Form

$$y'' + P(x)y' + U(x) = 0$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

## 3.4 Undetermined Coefficients

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Applied Anal.

$ay'' + by' + cy = g(x)$

To find a particular solution,

$y_p = \underset{g(x)}{\cdots}$ by the type of functions $g(x)$ - See table.

If the assumed $y_p$ is a solution of $ay'' + by' + cy = 0$

Then we assume a new particular solution $xy_p$

Notes (a) a, b, c, are constants

$x^2y'' + xy' + y = \sin x$ ✗

→ use variation of parameters

(b) $2y'' + 3y' - y = \dfrac{e^x}{(x+1)}$

→ use the variation of parameters

## 3.5 Variation of Parameters

Standard Form $y'' + P(x)y' + Q(x)y = f(x)$

Let $y = c_1y_1 + c_2y_2$ be the general solution of the associated homo. $y'' + P(x)y' + Q(x)y = 0$

A particular solution

$y_p = y_1u_1(x) + y_2u_2(x)$ For two functions $u_1(x)$, $u_2(x)$

$u_1' = \dfrac{W_1}{W}$, $u_2' = \dfrac{W_2}{W}$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$

$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

## 3.6 Cauchy-Euler Equations

$ax^2y'' + bxy' + cy = 0$

$\boxed{y = x^m}$ - Auxiliary Eq'n: $(am^2 + (b-a)m + c = 0)$

$\left\{ \text{Remember} : \quad \begin{matrix} ay'' + by' + cy = 0 & (y = e^{mx}) \\ ax^2y'' + bxy' + cy = 0 & (y = x^m) \end{matrix} \right\}$

Ex. $\rightarrow$ $x^2y'' - xy' + 3y = 0$

$m^2 + (-1-1)m + 3$

(I) $m_1 \neq m_2$ distinct real roots

$y = C_1 x^{m_1} + C_2 x^{m_2}$

(II) $m_1 = m_2$ repeated root

$y = C_1 x^{m_1} + C_2 x^{m_2} \ln x$

(III) $m_1 = \alpha + \beta i$, $m_2 = \alpha - \beta i$ are complex roots

$y = C_1 x^{\alpha} \cos(\beta \ln x) + C_2 x^{\alpha} \sin(\beta \ln x)$

Note: $\underline{x^2}y'' + \underline{x}y' + \underline{3}y = 2x^4e^x$

$\rightarrow$ use variation of parameters

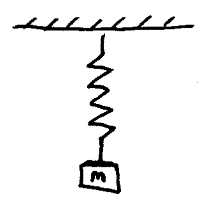
## 3.8 Linear Models

$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t)$

$m = ?$ $\beta = ?$ $k = ?$ $f(t) = \ldots$

initial conditions $x(0) = ?$

$x'(0) = ?$




## 4. Laplace Transform

4.1 $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

4.2 $\mathcal{L}^{-1}\{F(s)\} = f(t)$

if $\mathcal{L}\{f(t)\} = F(s)$

$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$

4.3 ✓ $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}\big|_{s \to s-a}$

$\mathcal{L}^{-1}\{F(s)\big|_{s\to s-a}\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$

2/ $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$

$\mathcal{L}^{-1}\{e^{-as}F(s)\}$

$\mathcal{L}^{-1}\{F(s)\} = f(t)$

4.4

1/ $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}$

2/ $\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$

$f * g(t) = \int_0^t f(y)\, g(t-y)\, dy$

$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}$

3/ If $f(t)$ is a periodic function with period $T$,

$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt$

4.5 The direct delta function

$\delta(t-t_0) = \lim_{a\to 0} \delta_a(t-t_0)$

$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$

$\mathcal{L}^{-1}\{e^{-st_0}\} = \delta(t-t_0)$

3.3 - homogeneous linear DE w/ constant coeff.

$ay'' + by' + cy = 0$ $\qquad y = e^{mx}$

where $a, b, c$ are constants

Auxiliary eq'n: $\boxed{am^2 + bm + c = 0}$

Case I : $m_1 \neq m_2$ ; real roots

$c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case II : repeated real root ; $m_1 = m_2$

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \cdot x$

Case III : $m_1 = \alpha + \beta i \qquad m_2 = \alpha - \beta i$

$y = e^{\alpha x} \cos(\beta x) + e^{\alpha x} \sin(\beta x)$

higher-order : $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$

$\boxed{y = e^{mx}}$

Auxiliary Equation : $a_n m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0 = 0$

(I) $m_1, m_2, \dots, m_n$ are distinct real roots

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$

(II) $m_1$ is a real root, repeated $k$ times

$e^{m_1 x}, \; e^{m_1 x} \cdot x, \; \dots, \; e^{m_1 x} \cdot x^{k-1}$

are $k$ linearly independent solutions.

(III) If $m_1 = \alpha + \beta i$ is complex root repeated $k$ times

then $m_2 = \alpha - \beta i$ ......

$e^{\alpha x} \cos(\beta x), \; x e^{\alpha x} \cos(\beta x), \; \dots, \; x^{k-1} e^{\alpha x} \cos(\beta x)$

$e^{\alpha x} \sin(\beta x), \; \dots\dots, \; x^{k-1} e^{\alpha x} \sin(\beta x)$

are $2k$ linearly indep solutions.