Nov. 20/17

Applied Anal.

$$\boxed{\begin{array}{l} \mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\} = e^{-as}\,\mathcal{L}\{f(t)\} \quad \text{or} \\ \mathcal{L}\{f(t)\,\mathcal{U}(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\} \end{array}}$$

$$\boxed{\begin{array}{l} \mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathcal{U}(t-a) \\ \text{Where } f(t) = \mathcal{L}^{-1}\{F(s)\} \end{array}}$$

Ex. $\mathcal{L}^{-1}\{e^{-\pi/2\, s}\frac{s}{s^2+a}\} = f(t-\pi/2)\,\mathcal{U}(t-\pi/2)$

where $f(t) = \mathcal{L}^{-1}\{\frac{s}{s^2+a}\} = \cos 3t$

$\mathcal{L}^{-1}\{e^{-\pi/2\, s}\frac{s}{s^2+a}\}\ \cos 3(t-\pi/2)\ \mathcal{U}(t-\pi/2)$

$= \cos(3t - 3\pi/2)\ \mathcal{U}(t-\pi/2)$

$= -\sin 3t \cdot \mathcal{U}(t-\pi/2) = \begin{cases} 0, & 0 \le t \le \pi/2 \\ -\sin 3t, & t > \pi/2 \end{cases}$

Ex. $\mathcal{L}\{\cos t \cdot \mathcal{U}(t-\pi)\}$

$= e^{-\pi s}\,\mathcal{L}\{\cos(t+\pi)\} = e^{-\pi s}\,\mathcal{L}\{-\cos t\}$

$= -e^{-\pi s}\frac{s}{s^2+1}$

Ex. Solve the IVP

$y' + y = f(t)$, $y(0) = 5$ where

$f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$

Solution: $f(t) = 3\cos t \cdot \mathcal{U}(t-\pi)$

$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{3\cos t\,\mathcal{U}(t-\pi)\}$

$sY(s) - y(0) + Y(s) = 3\cdot(-e^{-\pi s}\cdot\frac{s}{s^2+1})$ (above)

$(s+1)Y(s) = 5 - e^{-\pi s}\cdot\frac{3s}{s^2+1}$

$Y(s) = \frac{5}{s+1} - e^{-\pi s}\frac{3s}{(s+1)(s^2+1)}$

$y(t) = \mathcal{L}^{-1}\{\frac{5}{s+1}\} - \mathcal{L}^{-1}\{e^{-\pi s}\frac{3s}{(s+1)(s^2+1)}\}$

$= 5e^{-t} - f(t-\pi)\,\mathcal{U}(t-\pi)$

where $f(t) = \mathcal{L}^{-1}\{\frac{3s}{(s+1)(s^2+1)}\}$

$$\frac{3s}{(s+1)(s^2+1)} = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+1)}$$

$$\rightarrow \quad 3s = A(s^2+1) + Bs+C\,(s+1)$$

(1) $s = -1$: $\quad 3(-1) = A(1+1) + (Bs+C)\cancel{(-1+1)}^{\,0} \Rightarrow A = -3/2$

(2) constant term (s^0) : $\quad 0 = A + C \Rightarrow C = 3/2$

(3) constant term (s^2) : $\quad 0 = A + B \Rightarrow B = 3/2$

$$\Rightarrow \quad -3/2\, e^{-t} + 3/2 \cos t + 3/2 \sin t$$

$$y(t) = 5e^{-t} - \left[-3/2\, e^{-(t-\pi)} + 3/2 \cos(t-\pi) + 3/2 \sin(t-\pi) \right] \mathcal{U}(t-\pi)$$

$$\Rightarrow \begin{cases} 5e^{-t}, & 0 \le t < \pi \\ 5e^{-t} - \left[-3/2\, e^{-(t-\pi)} + 3/2 \cos(t-\pi) + 3/2 \sin(t-\pi) \right] & t \ge \pi \end{cases}$$

4.4 ADDITIONAL OPERATIONAL PROPERTIES

Nov. 22/17

APPLIED ANAL.

4.4.1 Derivation of transform

$$\frac{d}{ds}\,\mathcal{L}\{f(t)\} = ?$$

$$\frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{d}{ds} e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st}\cdot(-t)f(t)\,dt = -\int_0^\infty e^{-st}\, t\, f(t)\,dt$$

$$\boxed{\frac{d}{ds}\,\mathcal{L}\{f(t)\} = -\mathcal{L}\{t f(t)\}}$$

$$\frac{d^2}{ds^2}\,\mathcal{L}\{f(t)\} = \frac{d}{ds}\left(\frac{d}{ds}\,\mathcal{L}\{f(t)\}\right.$$

$$= \frac{d}{ds}\left(-\mathcal{L}\{t f(t)\}\right)$$

$$= -\frac{d}{ds}\,\mathcal{L}\{t f(t)\}$$

$$= -\left[-\mathcal{L}\{t(t f(t))\}\right]$$

$$= \mathcal{L}\{t^2 f(t)$$

Thm. 4.4.1 (Derivative of transform)

If $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots$

$$\mathcal{L}\{t^n f(t)\} = \frac{d^n}{ds^n}F(s)$$

$$\boxed{\cancel{\mathcal{L}\{t^n f(t)\} = \frac{d^n}{ds^n}\,\mathcal{L}\{f(t)\}\,(-1)^n}}$$

Ex. $\mathcal{L}\{t^2 \sin t\} = (-1)^2\left(\frac{d^2}{ds^2}\right)\mathcal{L}\{\sin t\}$

$$\boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\,\mathcal{L}\{f(t)\}}$$

$$= \frac{d^2}{ds^2}\cdot\frac{1}{s^2+1^2} = \frac{d}{ds}\cdot\frac{-2s}{(s^2+1)^2}$$

$$= -2(s^2+1)^2 + (-2s)(-2)(s^2+1)^{-3}\cdot(2s)$$

$$= \frac{-2}{(s^2+1)^2} + \frac{8s^2}{(s^2+1)^3} \;\ldots$$

Ex. $\mathcal{L}\{te^{2t}\} = (-1)^1 \frac{d}{ds}\mathcal{L}\{e^{2t}\}$
$= -\frac{d}{ds}\frac{1}{s-2} = \frac{1}{(s-2)^2}$

Ex. (a) Evaluate $\mathcal{L}\{t\sin 4t\}$
(b) Solve $x'' + 16x = \cos 4t$, $x(0) = 0$
$x'(0) = 1$

Solution a) $\mathcal{L}\{t\cdot\sin 4t\} = (-1)\frac{d}{ds}\mathcal{L}\{\sin 4t\}$
$= -\frac{d}{ds}\frac{4}{s^2+4^2} = -4\frac{d}{ds}(s^2+16)^{-1} = 4(s^2+16)^{-2}(2s)$
$\mathcal{L}\{t\sin 4t\} = \frac{8s}{(s^2+16)^2}$

b) $\mathcal{L}\{x''\} + 16\mathcal{L}\{x\} = \mathcal{L}\{\cos 4t\}$
$(s^2X(s) - sx(0) - x'(0)) + 16X(s) = \frac{s}{s^2+4^2}$
$(s^2+16)X(s) - 1 = \frac{s}{s^2+16}$
$X(s) = \frac{1}{(s^2+16)} + \frac{s}{(s^2+16)^2}$
$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{\frac{1}{s^2+16}\} + \mathcal{L}^{-1}\{\frac{s}{(s^2+16)^2}\}$
$\Rightarrow \mathcal{L}^{-1}\{\frac{1}{4}\cdot\frac{4}{s^2+4^2}\} + \mathcal{L}^{-1}\{\frac{1}{8}\cdot\frac{8s}{(s^2+16)^2}\}$
$= \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t$

4.4.2 Transform of Integrals

$\mathcal{L}^{-1}\{F(s)\,G(s)\} \neq \mathcal{L}^{-1}\{F(s)\}\,\mathcal{L}^{-1}\{G(s)\}$
$\mathcal{L}\{f(t)\,g(t)\} \neq \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$

$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}$
convolution!

$\mathcal{L}\{f * g(s)\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$
convolution

Definition: The convolution of $f(t)$ and $g(t)$ is the function defined by
$f*g(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$

$= \int_0^t f(y)\, g(t-y)\, dy$

Ex. If $f(t) = t^2$, $g(t) = t$ then

$$f * g(y) = \int_0^t f(y)\, g(t-y)\, dy$$
$$= \int_0^t y^2 (t-y)\, dy$$
$$= \int_0^t ty^2 - y^3\, dy$$
$$= t\, y^3/3 - y^4/4 \Big|_0^t$$
$$= t^4/3 - t^4/4 = 4t^4/12 - 3t^4/12 = 1/12\, t^4$$

Thm 4.4.2 (convolution thm.)

$$\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$$
$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}$$

Ex. $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s^2+1}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$, $\quad f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= f * g(t)$ $\quad = \sin t = g(t)$

$$= \int_0^t f(y)\, g(t-y)\, dy$$
$$= \int_0^t \sin y \,\sin(t-y)\, dy$$
$$= \int_0^t \tfrac{1}{2}\left[\cos(2y-t) - \cos(\cancel{y}+t-\cancel{y})\right] dy$$

$\sin A \sin B = 1/2\left[\cos(A-B) - \cos(A+B)\right]$

$$\Rightarrow \tfrac{1}{2}\left[\frac{\sin(2y-t)}{2} - y\cos t\right]\Big|_0^t$$
$$\Rightarrow 1/2\left[\left(1/2 \sin t - t\cos t\right) - \left(1/2 \sin(0-t) - 0\right)\right]$$
$$= 1/4 \sin t - 1/2\, t\cos t + 1/4 \sin t$$
$$= 1/2 \sin t - 1/2\, t\cos t$$

$g(t) = 1$, then $\mathcal{L}\{g(t)\} = 1/s$

$$f * g(t) = \int_0^t f(y)\, g(t-y)\, dy = \int_0^t f(y)\, dy$$
$$\mathcal{L}\left\{\int_0^t f(y)\, dy\right\} = \mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{1\}$$
$$= \frac{\mathcal{L}\{f(t)\}}{s}$$

$$\boxed{\mathcal{L}\left\{\int_0^t f(y)\, dy\right\} = \tfrac{1}{s}\,\mathcal{L}\{f(t)\}}$$

Nov. 24/17

Applied Anal.

Convolution of $f(t)$ and $g(t)$

$f * g(t) = \int_0^t f(y)\,g(t-y)\,dy$

$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g(t)$, where

$\mathcal{L}^{-1}\{F(s)\} = f(t)$, $\mathcal{L}^{-1}\{G(s)\} = g(t)$

Transform of integral:

$f(t)$ is a function, $g(t) = 1$

$f * g(t) = \int_0^t f(y)\,g(t-y)\,dy = \int_0^t f(y)\,dy$

$\mathcal{L}\{\int_0^t f(y)\,dy\} = \mathcal{L}\{f * g(y)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{1\}$

$$\boxed{\mathcal{L}\{\int_0^t f(y)\,dy\} = \tfrac{1}{s}\,\mathcal{L}\{f(y)\,dy\}}$$

$\mathcal{L}^{-1}\{\tfrac{1}{s}F(s)\} = \int_0^t f(y)\,dy$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$

Ex. $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}$

(where $F(s) = \frac{1}{s^2+1}$) $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= \sin t \qquad = f(t)$

$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin y\,dy$

$= -\cos y\,\Big|_0^t = -\cos t + 1$

Ex. $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot F(s)\right\}$

where $F(s) = \frac{1}{s(s^2+1)}$, $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t f(y)\,dy = \int_0^t 1 - \cos y\,dy$

$\Rightarrow\; y - \sin y\,\Big|_0^t = t - \sin t$

Example: Solve the integral equation

$f(t) = 3t^2 - e^{-t} - \int_0^t f(y)\,e^{t-y}\,dy$

Solution $\mathcal{L}\{f(t)\} = \mathcal{L}\{3t^2 - e^{-t}\} - \mathcal{L}\{f * g(t)\}$

Where $g(t) = e^t$ $(f * g(t) = \int_0^t f(y)\,g(t-y)\,dy$

$F(s) = 3\cdot\frac{2!}{s^{2+1}} - \frac{1}{s+1} - \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{e^t\}$

$F(s) = \frac{6}{s^3} - \frac{1}{s+1} - F(s)\cdot\frac{1}{s-1}$ ~

$$F(s)\left(1+\frac{1}{s-1}\right) = \frac{6}{s^3} - \frac{1}{s+1}$$

$$\left(\frac{s-1}{s-1}+\frac{1}{s-1}\right)$$

$$F(s)\left(\frac{s}{s-1}\right) = \frac{6}{s^3} - \frac{1}{s+1}$$

$$F(s) = \frac{6(s-1)}{s^4} - \frac{(s-1)}{(s)(s+1)}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{6(s-1)}{s^4} - \frac{s-1}{s(s+1)}\right\}$$

$$\Rightarrow 6\,\mathcal{L}^{-1}\left\{\frac{1}{s^3} - \frac{1}{s^4}\right\} - \mathcal{L}^{-1}\left\{\frac{s-1}{s(s+1)}\right\}$$

$$\Rightarrow 6\cdot\left(\tfrac{1}{2}\right)t^2 - \cancel{6}\left(\tfrac{1}{\cancel{6}}\right)t^3 - \mathcal{L}^{-1}\left\{\frac{s-1}{s(s+1)}\right\}$$

$$\Rightarrow \frac{s-1}{s(s+1)} = \frac{A}{s} + \frac{B}{(s+1)}$$

$$\Rightarrow s-1 = A(s+1) + B(s)$$

For $s=0$: $\quad 0-1 = A+0 \rightarrow A=-1$

For $s=-1$: $\quad -2 = 0 + -B \rightarrow B = 2$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{-1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s}\Big|_{s\to s+1}\right\}$$

$$\Rightarrow (-1) + (2)e^{-t}$$

$\Rightarrow$ then, $y = 3t^2 - t^3 + 1 - 2e^{-t}$

4.4.3 Transform of a periodic Function

Periodic Function with period $t>0$

$$f(t+T) = t \quad \text{for all } t$$

Ex. $f(t) = \sin t$ is a periodic Function with period 2π

$$\sin(2\pi + t) = \sin t \quad \text{for all } t$$

Thm. 4.4.3 If $f(t)$ is a piecewise continuous on $[0, +\infty)$ of exponential order, and periodic with period T, then

$$\boxed{\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt}$$

Proof: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^T e^{-st} f(t)\,dt + \boxed{\int_T^\infty e^{-st} f(t)\,dt \quad \begin{matrix} u = t - T \\ t = u + T \end{matrix}}$$

$$\boxed{= \int_0^T e^{-st} f(t)\,dt + \int_0^\infty e^{-s(u+T)} f(u+T)\,du}$$

$$\Rightarrow \int_0^T e^{-st} f(t)\,dt + \int_0^\infty e^{-su} \cdot e^{-sT} f(u)\,du$$

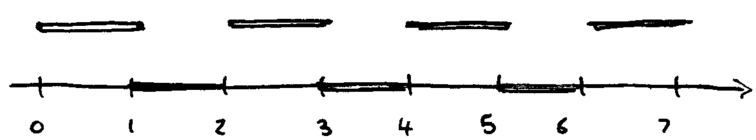
$$\mathcal{L}\{f(t)\} = \int_0^T e^{-sT} f(t)\,dt + e^{-sT} \int_0^\infty e^{-su} f(u)\,du$$

$$= \int_0^T e^{-sT} f(t)\,dt + e^{-sT}\, \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f(t)\}(1 - e^{-sT}) = \int_0^T e^{-st} f(t)\,dt$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Example: Find $\mathcal{L}\{E(t)\}$



Where $E(t)$ is a square wave

$E(t)$ is a periodic Function with period 2.

$$\mathcal{L}\{E(t)\} = \frac{1}{1-e^{-s\cdot 2}} \int_0^2 e^{-st} E(t)\,dt$$

$$\Rightarrow \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} \cdot 1\, dt + \int_1^2 \cancel{e^{-st} \cdot 0}\, dt \right]$$

$$\Rightarrow \frac{1}{1-e^{-2s}} \left[\frac{e^{-st}}{-s} \Big|_0^1 \right]$$

$$= \frac{1}{1-e^{-2s}} \left(\frac{-1}{s}\right)(e^{-s} - 1)$$

$$\Rightarrow \frac{(1-e^{-s})}{(s)(1-e^{-s})(1+e^{-s})}$$

$$\Rightarrow \frac{1}{s(1+e^{-s})}$$

$$\mathcal{L}\{E(t)\} = \frac{1}{(s)(1+e^{-s})}$$