Nov. 13/17

Applied Anal.

4.2.2 Transform of Derivatives

$$\boxed{\mathcal{L}\{f'(t)\} = -f'(0) + s\,\mathcal{L}\{f(t)\}}$$

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{(f'(t))'\}$$
$$= -f'(0) + s\,\mathcal{L}\{f'(t)\}$$
$$= -f'(0) + s\left[-f(0) + s\,\mathcal{L}\{f(t)\}\right]$$
$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$$

Thm. 4.2.2 If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, +\infty)$ and of exponential order and if $f^{(n)}$ is piecewise continuous on $[0, \infty)$ – then;

$$\mathcal{L}\{f^{(n)}t\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - f^{(n-1)}(0)$$

where $F(s) = \mathcal{L}\{f(t)\}$

Solve the linear DE's

Ex. use the laplace transform to solve

$dy/dx - 2y = 4\cos 2t$, $y(0) = 2$

Solution $\mathcal{L}\{y' - 2y\} = \mathcal{L}\{4\cos 2t\}$

$$\mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 4\,\mathcal{L}\{\cos 2t\}$$
$$\left[-y(0) + s\,\mathcal{L}\{y\}\right] - 2\mathcal{L}\{y\} = 4\cdot\frac{s}{s^2+4}$$
$$-2 + sY(s) - 2Y(s) = \frac{4s}{s^2+4}$$
$$sY(s) - 2Y(s) = 2 + \frac{4s}{s^2+4}$$
$$Y(s)(s-2) = 2 + \frac{4s}{s^2+4} = \frac{2(s^2+4) + 4s}{s^2+4}$$
$$Y(s) = \frac{2s^2 + 4s + 8}{(s-2)(s^2+4)}$$
$$y = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2s^2+4s+8}{(s-2)(s^2+4)}\right\}$$

$$\frac{2s^2 + 4s + 8}{(s-2)(s^2+4)} = \frac{A}{(s-2)} + \frac{Bs + C}{(s^2+4)}$$

$$2s^2 + 4s + 8 = A(s^2 + 4) + (Bs + c)(s - 2)$$

(1) $s = 2$; $8 + 8 + 8 = 8A + 0$; $A = 3$

(2) coeff. for s^2 ; $2 = A + B$; $B = -1$

(3) constant term ; $8 = 4A - 2C$; $C = 2$

$$\mathcal{L}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s-2} + \frac{-s+2}{s^2+4}\right\}$$

$$= 3\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + -1\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= 3e^{2t} - \cos 2t + \sin 2t$$

Ex. Solve the IVP Laplace Transform

$$y'' + 5y' - 14y = e^{-3t}, \quad y(0) = 1, \quad y'(0) = 2$$

Solution $\mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} - 14\mathcal{L}\{y\} = \mathcal{L}\{e^{-3t}\}$

$$[s^2Y(s) - sy(0) - y'(0)] + 5[sy(s) - y(0)] - 14Y(s) = \frac{1}{s+3}$$

$$s^2Y(s) - s - 2 + 5sY(s) - 5 - Y(s) = \frac{1}{s+3}$$

$$s^2Y(s) + 5sY(s) - 14Y(s) = s + 7 + 1/(s+3)$$

$$Y(s)(s^2 + 5s - 14) = s + 7 + \frac{1}{s+3} = \frac{(s+7)(s+3)+1}{s+3}$$

$$Y(s) = \frac{s^2 + 10s + 22}{(s+3)(s^2+5s-14)}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s^2 + 10s + 22}{(s+3)(s^2+5s-14)}\right\}$$

$$\frac{s^2 + 10s + 22}{(s+3)(s-2)(s+7)} = \frac{A}{(s+3)} + \frac{B}{(s-2)} + \frac{C}{(s+7)}$$

$$s^2 + 10s + 22 = A(s-2)(s+7) + B(s+3)(s+7) + C(s+3)(s-2)$$

- for $s = 2$; $4 + 20 + 22 = 45B$; $B = 46/45$
- For $s = -3$; $9 - 30 + 22 = -20A$; $A = -1/20$
- For $s = -7$; $49 - 70 + 22 = 36C$; $C = 1/36$

$$y(t) = \mathcal{L}^{-1}\left\{-1/20\,\frac{1}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{46}{45}\,\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{36} + \frac{1}{s+7}\right\}$$

$$= -\frac{1}{20}e^{-3t} + \frac{46}{45}e^{2t} + 1/36\,e^{-7t}$$

Thm. 4.2.3 (Behavior of $F(s)$ as $s \to \infty$)

If f is piecewise continuous on $[0, \infty)$ and of exponential order, then $\lim_{s \to \infty} \mathcal{L}\{f(t)\} = 0$

i.e. If $\mathcal{L}^{-1}\{F(s)\}$ exist, then

$$\lim_{s \to \infty} F(s) = 0$$

Example: Let $F(s) = \frac{s}{s+1}$: then,

$\lim_{s \to \infty} \frac{s}{s+1} = 1 \neq 0$. So $\mathcal{L}^{-1}\{F(s)\}$ DNE

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4.3 Translation Theorems

4.3.1 Translation on the s-axis

Thm 4.3.1 (First Translation Theorem)

If $\mathcal{L}\{f(t)\} = F(s)$, and a is any real number,

then; $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

Proof: $\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st}\left[e^{at} f(t)\right] dt$

$\Rightarrow \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a)$

Notation: $F(s-a) = F(s)\big|_{s\to s-a}$

$$\boxed{\mathcal{L}\{e^{at} f(t)\} = \mathcal{L}\{f(t)\}\big|_{s\to s-a}}$$

→ **Ex.** Use the first translation theorem to evaluate: $\mathcal{L}\{e^{2t} \cdot t^4\}$

$\Rightarrow \mathcal{L}\{t^4\}\big|_{s\to s-2}$

$\Rightarrow \dfrac{4!}{s^{4+1}}\Big|_{s\to s-2} \Rightarrow \dfrac{4\cdot 3\cdot 2\cdot 1}{s^5}\Big|_{s\to s-2} \Rightarrow \dfrac{24}{(s-2)^5}$

→ **Ex:** $\mathcal{L}\{e^{-3t} \sin 4t\} = \mathcal{L}\{\sin 4t\}\big|_{s\to s-(-3)}$

$\Rightarrow \dfrac{4}{s^2+16}\Big|_{s\to s+3} \Rightarrow \dfrac{4}{(s+3)^2+16}$

$\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)\big|_{s\to s-a}\}$

$\mathcal{L}^{-1}\{F(s)\big|_{s\to s-a}\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$

→ **Ex.** $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-2)^2}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\Big|_{s\to s-2}\right\}$

$= e^{2t}\mathcal{L}^{-1}\{1/s^2\} = e^{2t}\cdot t$

→ **Ex.** $\mathcal{L}^{-1}\left\{\dfrac{(2s+3)}{(s-5)^2}\right\}$ $\qquad \tilde{s} = s-5 \qquad s = \tilde{s}+5$

$\dfrac{2s+3}{\tilde{s}^2} = \dfrac{2(\tilde{s}+5)+3}{\tilde{s}^2}$

$\mathcal{C}$

$$= \mathcal{L}^{-1}\left\{\frac{2(s-5)+13}{(s-5)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2s+13}{s^2}\Big|_{s\to s-5}\right\}$$

$$\Rightarrow e^{5t}\cdot\mathcal{L}^{-1}\left\{\frac{2s+13}{s^2}\right\} = e^{5t}\mathcal{L}^{-1}\left\{\frac{2\cancel{s}}{s^{\cancel{2}}}\right\} + e^{5t}\mathcal{L}^{-1}\left\{\frac{13}{s^2}\right\}$$

$$\Rightarrow e^{5t}(2+13t)$$

→ Ex: $\mathcal{L}^{-1}\left\{\frac{(2s+7)}{s^2+6s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{2s+7}{(s+2)(s+4)}\right\}$

$$\Rightarrow \frac{2s+7}{(s+2)(s+4)} = \frac{A}{(s+2)} + \frac{B}{(s+4)} \quad \ldots$$

$$\mathcal{L}^{-1}\left\{\frac{2s+7}{s^2+6s+10}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2s+7}{(s+3)^2+1}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2(\tilde{s}-3)+7}{\tilde{s}^2+1}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{2\tilde{s}+1}{\tilde{s}^2+1}\right\}$$

$$s^2+6s+10$$
$$= s^2+2s\cdot 3+3^2+$$
$$= (s+3)^2+1$$

$$\tilde{s} = s+3 \to s = \tilde{s}-3$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+1}\Big|_{s\to s+3}\right\}$$

$$\Rightarrow e^{-3t}\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+1}\right\} = e^{-3t}\left[\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$$\Rightarrow e^{-3t}\left[2\cos t + \sin t\right]$$

→ Ex. Solve the IVP

$$y'' - 4y' + 4y = t^2e^{2t}, \quad y(0)=1, \quad y'(0)=1$$

Solution $\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{t^2e^{2t}\}$

$$[s^2Y(s) - s\,y(0) - y'(0)] - 4[sY(s) - y(0)] + 4Y(s) \ldots$$

$$\ldots = \mathcal{L}\{t^2 \,|_{s\to s-2}\}$$

$$s^2Y(s) - s - 1) - 4(sY(s) - 1) + 4(Y(s)) = \frac{2!}{s^{2+1}}\Big|_{s\to s-2}$$

$$s^2Y(s) - s - 1 - 4sY(s) + 4 + 4Y(s) = \frac{2}{(s-2)^3}$$

$$Y(s^2-4s+4) - s+3 = \frac{2}{(s-2)^3}$$

$$Y(s) = \frac{s-3}{(s^2-4s+4)} + \frac{1}{(s^2-4s+4)(s-2)^3}$$

$$Y(t) = \mathcal{L}^{-1}\left\{\frac{s-3}{(s-2)^2} + \frac{1}{(s-2)^5}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s-2)-1}{(s-2)^2} + \frac{1}{(s-2)^5}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s-1}{(s-2)^2} + \frac{1}{s^5}\Big|_{s \to s-2}\right\}$$

$$= e^{2t}\,\mathcal{L}^{-1}\left\{\frac{s-1}{s} + 1/s^5\right\}$$

$$= e^{2t}\left[\mathcal{L}^{-1}\left\{1/s - 1/s^2\right\} + \mathcal{L}^{-1}\left\{1/s^5\right\}\right]$$

$$= e^{2t}\left[1 - t + \mathcal{L}^{-1}\left\{\frac{4!}{s^{4+1}} \cdot \frac{1}{4!}\right\}\right]$$

$$= e^{2t}\left(1 - t + \frac{1}{4!} \cdot t^4\right)$$

Ex. Solve $y'' + 4y' + 5y = 3 + e^{-2t}$

$y(0) = 0, \quad y'(0) = 0$

Solution $\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{3 + e^{-2t}\}$

$$(s^2Y(s) - sY(0) - Y'(0)) + 4(sY(s) - Y'(0) + 5(Y(s))$$

$$s^2(Y(s)) + 4s(Y(s)) + 5(Y(s)) = 3/s + 1/(s+2)$$

$$Y(s)(s^2+4s+5) = 3/s + 1/(s+2) = \frac{3(s+2)+s}{s(s+2)}$$

$$Y(s) = \frac{4s+6}{s(s+2)(s^2+4s+5)}$$

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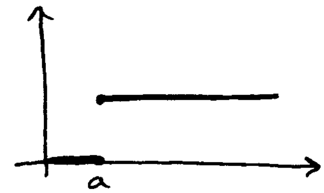
4.3.2 - Translation on the t-axis

$f(t-a)$?

Def. The unit step function

$$\mathcal{U}(t-a) = \begin{cases} 0 & \text{if } 0 \le t < a \\ 1 & \text{if } t \ge a \end{cases}$$



Ex $$f(t) = \begin{cases} 0 & \text{if } 0 \le t < 3 \\ \frac{1}{2}t & \text{if } t \ge 3 \end{cases} = \tfrac{1}{2}t \cdot \mathcal{U}(t-3)$$

$$1 - \mathcal{U}(t-a) = \begin{cases} 1 & \text{if } 0 \le t < a \\ 0 & \text{if } t \ge a \end{cases}$$

Ex. $$g(t) = \begin{cases} t^2, & 0 \le t < 3 \\ 0, & t \ge 3 \end{cases} = t^2 \cdot (1 - \mathcal{U}(t-3))$$

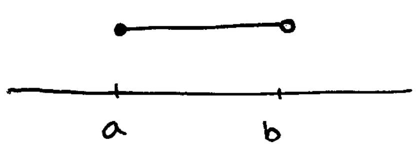
Ex. $$h(t) = \begin{cases} t^2, & 0 \le t < 3 \\ \frac{1}{2}t & t \ge 3 \end{cases}$$

$$\Rightarrow \begin{cases} t^2, & 0 \le t < 3 \\ 0 & t \ge 3 \end{cases} + \begin{cases} 0 & 0 \le t < 3 \\ \frac{1}{2}t & t \ge 3 \end{cases}$$

$$= t^2 (1 - \mathcal{U}(t-3)) + \tfrac{1}{2}t \cdot \mathcal{U}(t-3)$$

If $a < b$

$$\mathcal{U}(t-a) - \mathcal{U}(t-b) = \begin{cases} 0, & 0 \le t < a \\ 1, & a \le t < b \\ 0, & t \ge b \end{cases}$$



Ex. $f(t) = \begin{cases} 0 , & 0 \le t < 1 \\ e^t , & 1 \le t < 3 \\ 0 , & t \ge 0 \end{cases}$

$= e^t \left[\mathcal{U}(t-1) - \mathcal{U}(t-3) \right]$

$f(t-a)\,\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ f(t-a), & t \ge a \end{cases}$

Thm. 4.3.2 (Second Translation Theorem)

For $a > 0$, $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$

OR $\mathcal{L}\{f(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t-a)\}$

$\mathcal{L}\{f(t+a-a\}\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t+a\}$

PROOF : $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = \int_0^\infty e^{-st} f(t-a)\mathcal{U}(t-a)$

$= \int_0^a e^{-st} f(t-a)\,\mathcal{U}(t-a)\,dt + \int_a^\infty e^{-st} f(t-a)\,\mathcal{U}(t-a)$

$= \int_a^\infty e^{-st} f(t-a)\,dt \overset{}{\underset{u=t-a}{=\!=}} \int_0^\infty e^{-s(u+a)} f(u)\,du$

$= \int_0^\infty e^{-su} \cdot e^{-sa} f(u)\,du$

$= e^{-as}\int_0^\infty e^{-su} f(u)\,du = e^{-as}\mathcal{L}\{f(t)\}$

$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$

Ex. $\mathcal{L}\{\mathcal{U}(t-2)\} = e^{-2s}\mathcal{L}\{1\} = \frac{e^{-2s}}{s}$

Ex. $\mathcal{L}^{-1}\{\frac{1}{s-2} \cdot e^{-3s}\} = f(t-3)\mathcal{U}(t-3)$

where $f(t) = \mathcal{L}^{-1}\{\frac{1}{s-2}\} = e^{2t}$

$\mathcal{L}^{-1}\{\frac{1}{s-2} e^{-3s}\} = e^{2(t-3)}\mathcal{U}(t-3) = e^{2t-6}\,\mathcal{U}(t-3)$

$= \begin{cases} 0 , & 0 \le t < 3 \\ e^{2s-6} , & t \ge 3 \end{cases}$