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Applied Anal.

Pure resonance

Ex. Solve the IVP

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \nu t \;;\quad x(0) = 0 \;;\quad x'(0) = 0$$

restoring force   external force

Where ω, ν, F_0 are constants

Case (I), $\omega \neq \nu$

Solution Solve $x'' + \omega^2 x = 0$.

The general solution: $x_c = C_1 \cos(\omega t) + C_2 \sin(\omega t)$

To find a particular solution, assume

$$x_p = A\cos(\nu t) + B\sin(\nu t)$$

$$x_p'' = -A\nu^2 \cos(\nu t) - B\nu^2 \sin(\nu t)$$

$$[-A\nu^2 \cos(\nu t) - B\nu^2 \sin(\nu t)] + \omega^2[A\cos(\nu t) + B\sin(\nu t)] = F_0 \sin \nu t$$

$$(-A\nu^2 + A\omega^2)\cos(\nu t) + (-B\nu^2 + \omega^2 B)\sin(\nu t) = F_0 \sin(\nu t)$$

$$\begin{cases} -A\nu^2 + A\omega^2 = 0 \rightarrow A(-\nu^2+\omega^2) = 0 \rightarrow A = 0 \\ -B\nu^2 + \omega^2 B = F_0 \rightarrow B = \dfrac{F_0}{\omega^2 - \nu^2} \end{cases}$$

$$x_p = \frac{F_0}{\omega^2 - \nu^2} \sin(\nu t)$$

The general solution: $x = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \dfrac{F_0}{\omega^2 - \nu^2}\sin(\nu t)$

IVP: $x(0) = 0$; $x'(0) = 0$

$$x(0) = 0 \Rightarrow C_1 + 0 + 0 = 0 \rightarrow C_1 = 0$$

$$x(t) = C_2 \sin(\omega t) + \frac{F_0}{\omega^2 - \nu^2}\sin(\nu t)$$

$$x'(t) = C_2 \cos(\omega t) + \frac{F_0}{\omega^2 - \nu^2}\cdot \nu \cos(\nu t)$$

$$x'(0) \rightarrow C_2 \omega + \frac{F_0}{\omega^2 - \nu^2}\cdot \nu = 0$$

$$C_2 = \frac{-F_0 \nu}{\omega(\omega^2 - \nu^2)}$$

$$x = \frac{-F_0 \nu}{\omega(\omega^2 - \nu^2)}\sin(\omega t) + \frac{F_0}{\omega^2 - \nu^2}\sin(\nu t)$$

$$x(t) = \frac{F_0}{\omega(\omega^2 - \nu^2)}\left(-\nu \sin(\omega t) + \omega \sin(\nu t)\right)$$

if $\omega \neq \nu$

→ Case (II), $\omega = \nu$, Let $\nu \to \omega$

The Solution:

$$\lim_{\nu \to \omega} \frac{F_0(-\nu \sin(\omega t) + \omega \sin(\nu t))}{\omega(\omega^2 - \nu^2)}$$

$$\lim_{\nu \to \omega} \frac{d/d\nu \; F_0(-\nu \sin(\omega t) + \omega \sin(\nu t))}{d/d\nu \; \omega(\omega^2 - \nu^2)}$$

$$\lim_{\nu \to \omega} \frac{F_0(-\sin(\omega t) + \omega t \cos(\nu t))}{-2\nu\omega}$$

$$\Rightarrow \frac{F_0(-\sin(\omega t) + \omega t \cos(\omega t))}{-2\omega^2}$$

$$\Rightarrow \frac{F_0}{2\omega^2}\sin(\omega t) + \frac{-F_0 t}{2\omega}\cos(\omega t)$$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) + \frac{-F_0 t}{2\omega}\cos(\omega t)$$

If $t_n = \frac{n\pi}{\omega}$, $n = 1, 2, 3, \dots$, $\omega t_n = n\pi$

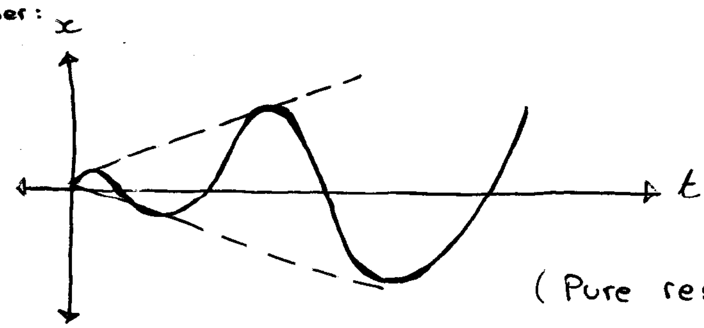
$$|x(t_n)| = \left|\frac{F_0}{2\omega^2}\sin(\omega t_n) + \frac{-F_0 t_n}{2\omega}\cos(\omega t_n)\right|$$

$$= \left|\frac{F_0}{2\omega^2}\sin(n\pi) + \frac{-F_0 t_n}{2\omega}\cos(n\pi)\right|$$

$$\Rightarrow \frac{|-F_0|\frac{n\pi}{\omega}}{2\omega} \qquad \cos n\pi = (-1)^n$$

$$\Rightarrow \frac{F_0}{2\omega^2}\pi \cdot n \to \infty \quad \text{as } n \to \infty$$

consider:



(Pure resonance)

Chapter 4: The Laplace Transform

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t)$$

4.1 - Definition of the Laplace Transform

Transform: an operation that transforms a function into another function.

Ex. $d/dx : f \to f'$

e.g. $d/dx \; x^2 \to 2x$

$d/dx \; \sin x \to \cos x$

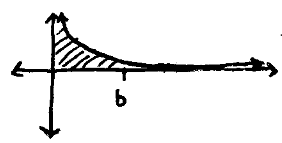
$\int : f \to f(x)\,dx$

$\int : x^2 \to x^3/3 + C$

Linearity of the transform:

$d/dx\,(\alpha f + \beta g) = \alpha \, d/dx\, f + \beta \, d/dx\, g$

Improper Integrals $\left(\int_0^\infty f(t)\,dt \right) = \lim_{b \to \infty} \int_0^b f(t)\,dt$



Definition: Let f be a function defined for $t \geq 0$. Then

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

provided the integral converges

$\mathcal{L}\{f(t)\}$ is a function of s (this is not a 5)

Notation $\mathcal{L}\{f(t)\} = F(s)$

$\mathcal{L}\{g(t)\} = G(s)$, $\mathcal{L}\{y(t)\} = Y(s)$

Example: $\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\,dt$

$\lim_{b\to\infty} \int_0^b e^{-st}\,dt$

$\Rightarrow \lim_{b\to\infty} -\frac{1}{s} e^{-st} \Big|_0^b$

$\Rightarrow \lim_{b\to\infty} (-1/s)(e^{-sb} - e^0)$

$\Rightarrow \lim_{b\to\infty} (-1/s)(\cancel{1/e^{sb}}^{\,0} - 1)$

$\Rightarrow (-1/s)(-1) \to 1/s$

$(s > 0)$

$$\boxed{\mathcal{L}\{1\} = 1/s, \quad s > 0}$$

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Applied Anal.

4.1 Laplace Transform

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 = 1/s \;,\; s > 0$

Example: $\mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t\,dt$

$= \lim_{b\to\infty} \int_0^b e^{-st} \cdot t\,dt \qquad (a = -s)$

$\int e^{at} \cdot t\,dt = (t)(\frac{1}{a} e^{at}) - \int (1/a\, e^{at})\,dt$

$\int uv - \int v \cdot (u'\,dt)$

$u = t, \quad dv = e^{at}\,dt \quad \ldots$ etc.

$\Rightarrow \lim_{b\to\infty} \frac{1}{-s} e^{-st}(t - \frac{1}{-s})\Big|_0^b$

$\Rightarrow \lim_{b\to\infty} [\frac{1}{-s} e^{-sb}(b + 1/s)] - [-\frac{1}{s} e^0 (0 + 1/s)]$

$\Rightarrow \lim_{b\to\infty} -1/s \; \frac{b + 1/s}{e^{sb}} + 1/s^2$

Where $\lim_{b\to\infty} \frac{b + 1/s}{e^{sb}} \; (\frac{\infty}{\infty}) = \lim_{b\to\infty} (\frac{1}{e^{sb}\cdot s}) = 0$

$\mathcal{L}\{t\} = 1/s^2$

Example: $\mathcal{L}\{e^{2t}\} = \int_0^\infty e^{-st} \cdot e^{2t}\,dt$

$= \lim_{b\to\infty} \int_0^b e^{-st} \cdot e^{2t}\,dt \qquad$ note $(e^\alpha \cdot e^\beta = e^{\alpha+\beta})$

$= \lim_{b\to\infty} \int_0^b e^{(-s+2)t}\,dt$

$= \lim_{b\to\infty} \frac{1}{-s+2} e^{(-s+2)t}\Big|_0^b$

$= \lim_{b\to\infty} \frac{1}{-s+2}(e^{(-s+2)b} - e^0) \qquad$ note $(e^\alpha = \frac{1}{e^{-\alpha}})$

$= \lim_{b\to\infty} \frac{1}{-s+2}(\frac{1}{e^{(s-2)b}} - 1) \qquad$ note $(s > 2)$

$= \frac{1}{-s+2}(0 - 1) = \frac{1}{s-2}, \qquad s > 2$

$\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}, \qquad s > 2$

Example: $\mathcal{L}\{\sin t\} = \int_0^\infty e^{-st} \sin t \, dt$

$= \lim_{b\to\infty} \int_0^b e^{-st} \sin t \, dt$

$\rightarrow \boxed{\int e^{at} \sin dt}_{X} \Rightarrow (\sin t)(\frac{1}{a}e^{at}) - \int \frac{1}{a} e^{at} \cos t \, dt$

$u = \sin t$

$dv = e^{at} \rightarrow v = \frac{e^{at}}{a}$

$= \frac{1}{a} \sin t \; e^{at} - \frac{1}{a} \int e^{at} \cos t \, dt \qquad \begin{matrix} U = \cos t \\ V = \frac{1}{a} e^{at} \end{matrix}$

$= \frac{1}{a} \sin t e^{at} - \frac{1}{a} \left[(\cos t)(\frac{1}{a} e^{at}) - \int (\frac{1}{a} e^{at})(-\sin t) dt \right.$

$= \frac{1}{a} \sin t e^{at} - \frac{1}{a^2} (e^{at} \cos t) - \frac{1}{a_2} \boxed{\int e^{at} \sin t \, dt}_{X}$

$X + \frac{1}{a^2} X = \frac{1}{a} e^{at} \sin t - \frac{1}{a^2} e^{at} \cos t = (1 + \frac{1}{a^2}) X$

$X = \frac{a^2}{a^2(1 + \frac{1}{a^2})} (\frac{1}{a} e^{at} \sin t - \frac{1}{a^2} e^{at} \cos t)$

$= \frac{a^2}{a^2+1} \cdot (\frac{1}{a}) e^{at} (\sin t - \frac{1}{a} \cos t)$

$= \frac{a}{a^2+1} e^{at} (\sin t - \frac{1}{a} \cos t)$

$\mathcal{L} = \lim_{b\to\infty} \int_0^b e^{-st} \sin t \, dt, \quad a = -s$

$= \lim_{b\to\infty} \frac{(-s)}{(-s)^2+1} e^{-st} (\sin t - \frac{1}{-s} \cos t) \Big|_0^b$

$= \lim_{b\to\infty} \frac{s}{s^2+1} \left[e^{-sb} (\sin b + (\frac{1}{s}) \cos b - e^0 (0 + \frac{1}{s}) \right]$

$= \frac{-s}{s^2+1} \left[0 - \frac{1}{s} \right] = \frac{1}{s^2+1}, \quad s > 0$

Where $\lim_{b\to\infty} \frac{\sin b + \frac{1}{s} \cos b}{e^{sb}} = 0$

$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

Theorem 4.1.1

a) $\mathcal{L}\{1\} = \frac{1}{s}$

b) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \dots$

c) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

d) $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$

e) $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$

f) $\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$

g) $\mathcal{L}\{\cosh kt\} \quad \frac{s}{s^2-k^2}$

were $\begin{cases} \sin zh = \frac{1}{2}(e^z - e^{-z}) \\ \cos zh = \frac{1}{2}(e^z + e^{-z}) \end{cases}$

$\mathcal{L}\{\alpha f(t) + \beta g(t)\}$

$\Rightarrow \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{f(t)\}$

<u>Example</u> $\mathcal{L}\{2 + 3t^2 + e^{2t} - 5\sin t\}$

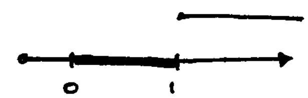
$= \mathcal{L}\{2\} + \mathcal{L}\{3t^2\} + \mathcal{L}\{e^{2t}\} + \mathcal{L}\{5\sin t\}$

$= 2\,\mathcal{L}\{1\} + 3\,\mathcal{L}\{t^2\} + \mathcal{L}\{e^{2t}\} - 5\,\mathcal{L}\{\sin t\}$

$= 2 \cdot \frac{1}{s} + 3 \cdot \frac{2!}{s^{2+1}} + \frac{1}{s-2} - 5 \cdot \frac{1}{s^2+1}$ (by formulas)

<u>Example</u>: Evaluate $\mathcal{L}\{f(t)\}$ for

$$f(t) = \begin{cases} 0 & \le t < 1 \\ 1, & t \ge 1 \end{cases}$$



Solution: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\,dt$

$= \lim_{b\to\infty} \int_0^b e^{-st} f(t)\,dt = \lim_{b\to\infty} \int_0^1 e^{-st} \cdot 0\,dt + \int_1^b e^{-st} \cdot 1\,dt$

$= \lim_{b\to\infty} \frac{1}{-s} e^{-st}\Big|_1^b = \lim_{b\to\infty} \frac{1}{-s}(e^{-sb} - e^{-s})$

$= \lim_{b\to\infty} \frac{1}{-s}\left(\frac{1}{e^{sb}} - e^{-s}\right) = -\frac{1}{s}(0 - e^{-s})$

$= \frac{1}{s} e^{-s}$

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Applied Analysis

4.1 Laplace Transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

Sufficient conditions for existance of $\mathcal{L}\{f(t)\}$

<u>Definition:</u> A function $f(t)$ is said to be of exponential order c if there exist constants c, M, T, such that

$$|f(t)| \le M e^{ct}, \quad \text{for all } t \ge T$$

<u>Ex</u> (1) $f(t) = t$ is of exponential 1

$|t| \le e^t$ for all $t \ge 0$

(2) $f(t) = \sin t$ is of exponential orders

$|\sin t| \le e^t$ for all $t \ge 0$

Thm. 4.1.2 If $F(t)$ is piecewize continuous on $[0, +\infty]$ and of exponential order c, then $\mathcal{L}\{f(t)\}$ exists for $s > c$.

4.2 Inverse Transform

$$\begin{tikzcd} f(t) \arrow[r, "\mathcal{L}"] & \mathcal{L}\{f(t)\} = F(s) \arrow[l, bend left, "\mathcal{L}^{-1}"] \end{tikzcd}$$

$$\begin{tikzcd} f(t) \arrow[r, "\text{derivatives}"] & f'(t) \arrow[l, bend left, "\int"] \end{tikzcd}$$

<u>Definition</u> If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

$f(t)$ is the inverse Laplace Transform of $F(s)$

Example $\mathcal{L}^{-1}\{3/s\} = 3\mathcal{L}^{-1}\{1/s\} = 3$

Ex. $\mathcal{L}^{-1}\{1/s^2\} = \mathcal{L}^{-1}\{6!/s^{6+1}\}\ 1/6!$

$= 1/6!\ t^6 = \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} t^6 = \frac{1}{720} t^6$

Ex. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\}\cdot\frac{1}{\sqrt{2}}$

$= 1/\sqrt{2}\ \sin(2t)$

Ex. $\mathcal{L}^{-1}\left\{\frac{5s-3}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{5s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{-3}{s^2+9}\right\}$

$= 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$

$= 5\cos 3t - \sin 3t$

Ex. $\mathcal{L}^{-1}\left\{\frac{s^2-2s+9}{(s-2)(s+3)(s+7)}\right\}$

$\int \frac{x^2-2x+9}{(x-2)(x+3)(x+7)}\,dx$

$$\frac{x^2-2x+9}{(x-2)(x+3)(x+7)} = \frac{A}{(x-2)} + \frac{B}{(x+3)} + \frac{C}{(x+7)}$$

$s^2 - 2x + 9 = A(x+7)(x+3) + B(x-2)(x+7) + C(x-2)(x+3)$

(1) $s = 2$: $4 - 4 + 9 = A(2+3)(2+7) + 0 + 0$

$9 = A\cdot 5\cdot 9,\quad A = 1/5$

(2) $s = -3$: $(-3)^2 - 2(-3) + 9 = 0 + B(-3-2)(-3+7) + 0$

$24 = B(-5)4,\quad 6 = -5B,\quad B = -6/5$

(3) $s = -7$: $(-7)^2 - 2(-7) + 9 = 0 + 0 + C(-7-2)(-7+3)$

$49 + 14 + 9 = C(-9)(4)$

$72 = 36C,\quad C = 2$

$\mathcal{L}^{-1}\left\{\frac{1/5}{s-2} + \frac{-6/5}{s+3} + \frac{2}{s+7}\right\}$

$\Rightarrow 1/5\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 6/5\,\mathcal{L}^{-1}\left\{\frac{1}{s-(-3)}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-(-7)}\right\}$

$\Rightarrow 1/5\,e^{2t} - 6/5\,e^{-3t} + 2e^{-7t}$

Ex. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s-10}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s+5)}\right\}$

$$\frac{1}{(s-2)(s+5)} = \frac{A}{(s-2)} + \frac{B}{(s+5)}$$

$$1 = (s+5)A + (s-2)B$$

(1) $s = 2$: $1 = (2+5)A + 0$; $A = 1/7$

(2) $s = -5$: $1 = 0 + (-5-2)B$; $B = -1/7$

$$\mathcal{L}^{-1}\{1/7 \cdot 1/s-2\} + \mathcal{L}^{-1}\{-1/7 \cdot 1/s+5\}$$

$$= 1/7\, \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 1/7\, \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$= 1/7\, e^{2t} - 1/7\, e^{-5t}$$

4.2.2 Transform of Derivatives

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$ $\qquad \int u\,dv$

$= \lim_{b\to\infty} \int_0^b \underbrace{e^{-st}}_{u}\, \underbrace{f'(t)\,dt}_{dv}$ $\quad v = f(t)$ $\qquad = uv - \int v\,du$

$$= \lim_{b\to\infty} \left(e^{-st} f(t)\right) - \int f(t)(-s)e^{-st}\,dt \Big|_0^b$$

$$= \lim_{b\to\infty} e^{-sb} f(b) - e^0 f(0) + s\int_0^b e^{-st} f(t)\,dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,\mathcal{L}\{f(t)\}$$

where $\lim_{b\to\infty} e^{-sb} f(b) = \lim_{b\to\infty} \frac{f(b)}{e^{sb}} = 0$

$$\mathcal{L}\{f'(t)\} = -f(0) + s\,\mathcal{L}\{f(t)\}$$