Chapter 1 - Introduction

1.1 Definitions and Terminology

Equation $2x+3=0$, x = unknown number

$$\frac{dy}{dx} = 6x^2y$$

x = variable

y = function of x

Find y: $\boxed{y = e^{2x^3} \text{ is a solution}}$

How do you know $y = e^{2x^3}$ is a solution of $dy/dx = 6x^2y$?

How do you know $x = -3/2$ is a solution of $2x+3=0$?

LHS: $2x+3$: $\cancel{2}(-3/\cancel{2}) + 3 = 0$

RHS: 0 = LHS, ∴ yes, its a solution of the DE

LHS: $dy/dx = e^{2x^3} \cdot 6x^2$

RHS: $6x^2y = 6x^2e^{2x^3}$ = LHS, ∴ yes, its a solution of the DE

Differential Equation (DE): the equation containing the derivatives of one or more dependent variables w.r.t. one or more independent variables.

Example (1) $dy/dx = \cos x$ (O) - 1st - linear

y = antiderivative of $\cos x$

$y = \int \cos x \, dx = \sin x$

(2) $\frac{d^2y}{dx^2} + 3y = 0$ (O) - 2nd - linear

(3) $\frac{\partial u}{\partial t} = h^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ (P) - 2nd - linear

$$\frac{\partial u}{\partial t} = \frac{du}{dt}$$

(4) $L\dfrac{d^2 i}{dt^2} + R\dfrac{di}{dt} + \dfrac{1}{C} i = E\omega \cos(\omega t)$ (o) - 2nd - linear

Where L, R, C, E, ω are constants

(5) $\dfrac{d^2V}{dx^2} + \dfrac{d^2V}{dy^2} = 0$ (p) - 2nd - linear

(6) $\dfrac{d^3x}{dy^3} + x\dfrac{dx}{dy} - 4xy = 0$ (o) - 3rd - non-linear

$y = 1/x$, $x = 1/y$

(7) $\dfrac{d^2y}{dx^2} + 7\left(\dfrac{dy}{dx}\right)^3 - 8y = 0$ (o) - 2nd - non-linear

(8) $\dfrac{d^2y}{dx^2} + \dfrac{d^2x}{dt^2} = x$ (o) - 2nd - linear

$\rightarrow (y'' + x'' = x)$

Classification

(1) by type : ordinary DE
(only one independent variable)
partial (two or more independent variables)

(2) by order : the highest order of the derivative

(3) by linearity : the dependent variable y and all its derivatives are of the first degree

$$\left[\begin{array}{l} 2x + 3 = 0 \rightarrow \text{linear} \\ 2x^2 - 3x + 5 = 0 \rightarrow \text{non-linear} \end{array}\right]$$

Example : $y' + 2y - x^2 = 0$
is a linear diff. equation

$\rightarrow y' - 2y^2 - x = 0$
is non-linear

Solution: Functions that satisfy the DE.

Domain of the ~~Functions~~ solutions **must be an interval**

Example: Domain of $y = 1/x$ as a Function is $(-\infty, 0) \cup (0, \infty)$

If $y = 1/x$ is a solution of a DE, then the domain of the solution will be either $(-\infty, 0)$ or $(0, \infty)$

Example: Verify that $y = \sqrt{x}$ is a solution of $\boxed{dy/dx = x/2y^3}$

And find the domain of the solution.

Solution: LHS $dy/dx = d/dx \sqrt{x} = \frac{1}{2} x^{-1/2}$

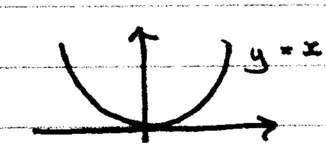
RHS $x/2y^3 = x/2(\sqrt{x})^3 = \dfrac{x}{2(\sqrt{x})^2 \sqrt{x}} = \dfrac{1}{2} \cdot x^{-1/2}$ $= LHS$

Domain of the solution: $(0, +\infty)$

Solution Curve: The graph of a solution

Ex. $y = x^2$ is a solution of $dy/dx = 2x$

The solution curve of $y = x^2$ is the graph of the function



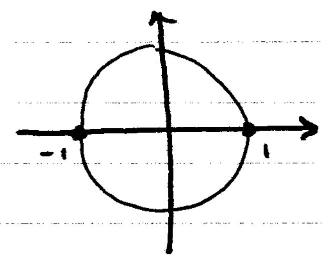
Explicit Solution: Can be written as $y = f(x)$.

Implicit Solution: A solution that is an implicit function (usually defined by a relation $G(x, y) = 0$)

Ex. $y = -x$, $x + y = 0$
(explicit) (implicit)

Ex. y is an implicit Function of x defined by $y^2 + x = e^{xy} + \cos y$.

Ex. Verify that, for $-1 < x < 1$, the relation $x^2 + y^2 - 1 = 0$ is an implicit Solution of $\dfrac{dy}{dx} = \dfrac{-x}{y}$



Solution: (implicit differentiation)

$\rightarrow$ $d/dx\,(x^2+y^2-1) = d/dx\,0$

$\rightarrow$ $d/dx\,x^2 + d/dx\,y^2 - d/dx\,1 = 0$

$2x + d/dx\,(y^2) - 0 = 0$

$2x + dy^2/dy \cdot dy/dx = 0$

$\cancel{2}y\,\dfrac{dy}{dx} = -\cancel{2}x$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

[Chain Rule:
$d/dx\,F(u) = \dfrac{dF(u)}{du} \cdot \dfrac{du}{dx}$]

How many solutions can a DE have?

(a) None: e.g. $\left(\dfrac{dy}{dx}\right)^2 + 1 = 0$

(b) Unique:

$\left(\dfrac{dy}{dx}\right)^2 + y^2 = 0$ has a unique Solution $y = 0$

(c) In most cases, there are infinitely many Solutions!

Example: Solve $dy/dx = 2x$

Solution: y - anti-derivatives of $2x$

$$y = \int 2x\,dx = x^2 + C$$

$C = 0$; $y = x^2$

$C = 1$; $y = x^2 + 1$

$C = 100$; $y = x^2 + 100$

$C = -5$; $y = x^2 + 5$

... (infinite many)

$y = x^2 + C$ – a family of solutions of one arbitrary parameter

Ex. Verify that for each pair of constants, C_1 and C_2, $\boxed{y = C_1 \cos 4x + C_2 \sin 4x}$ is a solution of $y'' + 16y = 0$

(- 2-parameter family of solutions)

Solution: LHS $= y'' + 16y$

$= (C_1 \cos 4x + C_2 \sin 4x)'' + 16((C_1 \cos 4x) + (C_2 \sin 4x))$

$= (-4C_1 \sin 4x + 4C_2 \cos 4x)' + 16(C_1 \cos 4x + C_2 \sin 4x)$

$= -16C_1 \cos 4x - 16C_2 \sin 4x + 16(C_1 \cos 4x + C_2 \sin 4x)$

$= 0 =$ RHS

Particular Solution: a solution of a DE that is free arbitrary ~~Process~~ parameter

Ex. Verify that $y = (x^2/4 + c)^2$ is a 1-parameter family of solutions of $\frac{dy}{dx} = xy^{1/2}$ and $y = \frac{x^4}{16}$ is a particular solution.

Solution: LHS $= dy/dx = d/dx\,(x^2/4 + c)^2$
$= 2\cdot(x^2/4 + c)\cdot 2x/4 = x(x^2/4 + c)$

RHS $= xy^{1/2} = x\,[(x^2/4 + c)^2]^{1/2}$
$= x\,(x^2/4 + c)$
$=$ LHS

$c = 0$, $y = (x^2/4 + 0)^2 = \left(\frac{x^2}{4}\right)^2 = x^4/16$

is a particular solution.