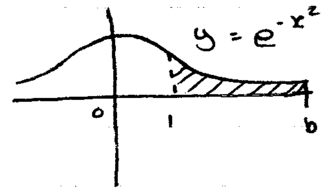
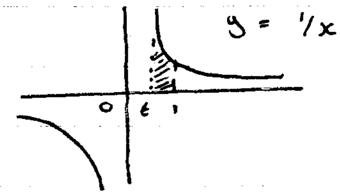
Improper Integrals (section 8.8)

$$\int_1^{\infty} e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx$$



$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx$$



Definition :

Improper integrals with infinite integration limits

Let f be a continuous function

Then,

1. $\displaystyle\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$

2. $\displaystyle\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$

3. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$

   where c is any real number.

   Converges if both $\int_{-\infty}^{c} f(x)\,dx$ and $\int_c^{\infty} f(x)\,dx$ converge. Otherwise it diverges.

Example :

1. $\displaystyle\int_0^{\infty} xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx$

$$\int xe^{-x}\,dx = -xe^{-x} - \int(-e^{-x})\,dx$$
$$= -xe^{-x} - e^{-x} + C$$

$u = x$	$\Rightarrow$	$u' = 1$
$v' = e^{-x}$	$\Rightarrow$	$v = -e^{-x}$

$$\int_0^b xe^{-x}\,dx = (-xe^{-x} - e^{-x})\Big|_0^b$$
$$= -be^{-b} - e^{-b} + 1$$

$$\lim_{b\to\infty} e^{-b} = \lim_{b\to\infty}\frac{1}{e^b} = 0$$

$$\lim_{b\to\infty} be^{-b} = \lim_{b\to\infty} \frac{b}{e^b} = \lim_{b\to\infty} \frac{1}{e^b} = 0$$

$$= 1$$

2. $$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_0^{\infty} xe^{-x^2}dx + \int_{-\infty}^{0} xe^{-x^2}dx$$

$$\int_0^{\infty} xe^{-x^2}dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}dx = \lim_{b\to\infty}\left(-\tfrac{1}{2}e^{-x^2}\Big|_0^b\right)$$

$$\int_{-\infty}^{0} xe^{-x^2}dx = \lim_{a\to-\infty}\int_a^0 xe^{-x^2}dx = \lim_{a\to-\infty}\left(-\tfrac{1}{2}e^{-x^2}\Big|_a^0\right)$$

$$\lim_{b\to\infty}\left(-\tfrac{1}{2}e^{-b^2}+\tfrac{1}{2}\right) = \tfrac{1}{2}$$

$$\lim_{a\to-\infty}\left(-\tfrac{1}{2}+\tfrac{1}{2}e^{-a^2}\right) = -\tfrac{1}{2}$$

3. $$\int_{-\infty}^{0}\frac{1}{1+x^2}dx = \lim_{a\to-\infty}\int_a^0\frac{1}{1+x^2}dx = \lim_{a\to-\infty}\left(\arctan x\Big|_a^0\right)$$

$$= \lim_{a\to-\infty}\left(\arctan 0 - \arctan a\right)$$

$$= \lim_{a\to-\infty} -\arctan a = -(-\pi/2) = \pi/2 \quad \text{convergent}$$

Definition (Improper Integrals with Infinite Discontinuities)

1. Let f be continuous on $(a, b]$ that has an infinite discontinuity at a $\left(\lim_{x\to a^+} f(x) = \infty \text{ or } -\infty\right)$ Then.

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

a ← t   b

2. - - - - - - - - - - - $[a, b)$ - - - - - - -
- - - - - - - - - - - - $\left(\lim_{x\to b^-} f(x) = \infty \text{ or } -\infty\right)$ Then.

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

a   t →   b

3. Let f be a function that is continuous on $[a, b]$ except at c in $[a, b]$, where it has an infinite discontinuity.

$$\int_a^b f(x)\,dx = \int_a^{c} f(x)\,dx + \int_{c}^b f(x)\,dx$$

Examples

1. $$\int_0^1 1/\sqrt{x}\,dx = \lim_{t\to 0^+} \int_t^1 1/\sqrt{x}\,dx = \lim_{t\to 0^+} (2 - 2t^{1/2}) = 2$$

$$\int 1/\sqrt{x}\,dx = \int x^{-1/2}\,dx = 2x^{1/2} + c$$

$$\int_t^1 \frac{1}{\sqrt{x}}\,dx = 2x^{1/2}\Big|_t^1 = 2 - 2t^{1/2}$$

2. $$\int_{-1}^1 1/x\,dx = \int_0^1 1/x\,dx + \int_{-1}^0 1/x\,dx \quad \text{(Divergent)}$$

$$\int_0^1 1/x\,dx = \lim_{t\to 0^+} \int_t^1 1/x\,dx = \lim_{t\to 0^+} \ln|x|\Big|_t^1 = \infty \quad \text{(Diverges)}$$

$$\int_{-1}^0 1/x\,dx = \lim_{t\to 0^-} \int_{-1}^t 1/x\,dx = \lim_{t\to 0^-} \ln|x|\Big|_{-1}^t = \infty \quad \text{(Diverges)}$$

- Trigonometric substitution (sec. 8.4)
- Partial Fractions (sec. 8.5)

$\sqrt{a^2 - x^2}$ $\quad x = a\sin t$

$a > 0$

$\sqrt{a^2 + x^2}$ $\quad x = a\tan t$

$\sqrt{x^2 - a^2}$ $\quad x = a\sec t$

1 $$\int \frac{dx}{x^2\sqrt{25 - x^2}} = \int \frac{5\cos t \, dt}{25\sin^2 t\sqrt{25 - 25\sin t}}$$

$x = 5\sin t$

$dx = 5\cos t \, dt$

$$\Rightarrow \int \frac{5\cos t}{25\sin^2 t\sqrt{25\cos^2 t}} dt$$

$$\Rightarrow \frac{5\cancel{\cos t}}{25\sin^2 t \, 5\cancel{\cos t}} dt$$

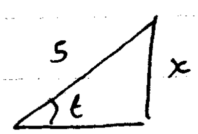
$$\rightarrow \frac{1}{25}\int \frac{1}{\sin^2 t} dt$$

$$\Rightarrow \frac{1}{25}\int \csc^2 t \, dt$$

$$= \frac{-1}{25}\cot + C$$

$$= \frac{-1}{25}\frac{\sqrt{25 - x^2}}{x} + C$$

$\sin t = \frac{x}{5}$



$\cot t = \frac{\sqrt{25 - x^2}}{x}$

(2) $\displaystyle\int \frac{x}{\sqrt{x^2+6x+12}}\,dx = \int \frac{x}{\sqrt{(x+3)^2+(\sqrt{3})^2}}$

$x+3 = \sqrt{3}\tan t$

$x = \sqrt{3}\tan t - 3$

$dx = \sqrt{3}\sec^2 t\,dt$

$$\boxed{\begin{aligned} &x^2+6x+12 \quad +9-9 \\ &x^2+6x+3 \quad +9 \\ &(x+3)^2+3 \end{aligned}}$$

$$\int \frac{x}{\sqrt{(x+3)^2+(\sqrt{3})^2}}\,dx = \int \frac{\sqrt{3}\tan t - 3}{\sqrt{3\tan^2 t+3}} \cdot \sqrt{3}\sec^2 t\,dt$$

$$= \int \frac{\sqrt{3}\tan t - 3}{\sqrt{3\sec^2 t}} \cdot \sqrt{3}\sec^2 t\,dt \qquad \sec t \geq 0$$

$$= \int \frac{\sqrt{3}\tan t - 3}{\sqrt{3}\sec t} \cdot \sqrt{3}\sec^2 t\,dt$$

$$= \int (\sqrt{3}\tan t - 3)\sec t\,dt = \sqrt{3}\int \tan t\sec t\,dt - 3\int \sec t\,dt$$

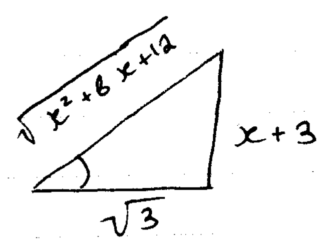
$$= \sqrt{3}\sec t - 3\ln\left|\sec t + \tan t\right| + C$$

$$= \frac{\sqrt{x^2+6x+12}}{\sqrt{3}} - \ln\left|\frac{\sqrt{x^2+6x+12}}{\sqrt{3}} + \frac{x+3}{\sqrt{3}}\right| + C$$

$x+3 = \sqrt{3}\tan t$

$\tan t = \dfrac{x+3}{\sqrt{3}}$

$\sec t = ?$



$$\sec t = \frac{\sqrt{x^2+6x+12}}{\sqrt{3}}$$

(3) $$\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx = \int_0^{\pi/6} \frac{\sqrt{3\sec^2 t - 3}}{\sqrt{3}\sec t} \cdot \sqrt{3}\sec t \tan t\, dt$$

$$x = \sqrt{3}\sec t$$
$$dx = \sqrt{3}\sec t \tan t\, dt$$

$$x = 2 \Rightarrow 2 = \sqrt{3}\sec t \Rightarrow \cos t = \sqrt{3}/2 \Rightarrow \pi/6$$
$$x = \sqrt{3} \Rightarrow \sqrt{3} = \sqrt{3}\sec t \Rightarrow \cos t = 1 \Rightarrow \emptyset$$

$$= \int_0^{\pi/6} \frac{\sqrt{3\tan^2 t}}{\cancel{\sqrt{3}\sec t}} \cdot \cancel{\sqrt{3}\sec t} \cdot \tan t\, dt = \int_0^{\pi/6} \sqrt{3}\tan^2 t\, dt$$

$$= \sqrt{3}\int_0^{\pi/6} (\sec^2 t - 1)\, dt$$

(consider: $\tan^2 t + 1 = \sec^2 t$)

$$= \sqrt{3}\int_0^{\pi/6} (\sec^2 t - 1)\, dt$$

$$= \sqrt{3}(\tan t - t)\Big|_0^{\pi/6} = \sqrt{3}(\tan \pi/6 - \pi/6) = \sqrt{3}\left(\frac{\sqrt{3}}{3} - \frac{\pi}{6}\right)$$

Partial Fractions

(1) $$\int \frac{3x^3 - 2x^2 + 1}{x^2 - 3x + 2}\, dx$$

$$\begin{array}{r} 3x + 7 \\ x^2 - 3x + 2 \overline{)\, 3x^3 - 2x^2 + 1} \\ -(3x^3 - 9x^2 + 6x) \\ \hline \emptyset \quad 7x^2 + 6x + 1 \\ -(7x^2 - 21x + 14) \\ \hline 15x - 13 \end{array}$$

$$= \int \left(3x + 7 + \frac{15x - 13}{x^2 - 3x + 2}\right) dx$$

$$= \frac{3x^2}{2} + 7x + \int \frac{15x - 13}{x^2 - 3x + 2}\, dx$$

$$\frac{15x - 13}{x^2 - 3x + 2} = \frac{15x - 13}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$x - 1 \rightsquigarrow \frac{A}{x-1}$$

$$x - 2 \rightsquigarrow \frac{B}{x-2}$$

$$\Rightarrow 15x - 13 = A(x-2) + B(x-1)$$
$$= (A+B)x - 2A - B$$

$\Rightarrow A + B = 15$ ①

$\Rightarrow 2A + B = 13$ ②

② − ① $\Rightarrow A = -2 \Rightarrow B = 17$

$$\int \frac{15x - 13}{x^2 - 3x + 2}\, dx = \int \left(\frac{-2}{x-1} + \frac{17}{x-2} \right) dx =$$

$$\Rightarrow -2 \ln|x-1| + 17 \ln|x-2| + C$$

Lecture Sequences (Section 9.1)

Sequence:

$a_1, a_2, a_3, \ldots$

$a_n = \frac{(-1)^n}{n}, \quad n = 1, 2, 3 \ldots$

$$\begin{array}{ccccc} a_1, & a_2, & a_3, & a_4 & \ldots \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel & \\ -1 & 1/2 & -1/3 & 1/4 & \ldots \end{array}$$

Equivalently

$f \mathbb{N} - \mathbb{R}$

$f(n) = a_n$

Def'n

A sequence is a function if: $\mathbb{N} \Rightarrow \mathbb{R}$ from the positive numbers to the real integers.

The numbers, $a_1 = f(1)$, $a_2 = f(2)$, ...

are the terms of the sequence.

The number a_n is the n^{th} term of the sequence. The entire sequence is denoted by

$\{a_n\}$ ($\{a_n\}_{n \in \mathbb{N}}$, (a_n), $(a_n)_{\in \mathbb{N}}$)

$(a_n)_{n=1}^{\infty}$

Def'n (limit of a seq.)

We say that the seq. $\{a_n\}$ has limit L if for every $\varepsilon > 0$ there $N \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \quad \text{for all } n > N$$

Thm (Properties of limits)

If $\lim_{n \to \infty} a_n = L$ and $\lim_{n \to \infty} b_n = K$

(1) $\lim_{n \to \infty} (a_n \pm b_n) = L \pm K$

(2) $\lim_{n \to \infty} c a_n = c \lim_{n \to \infty} a_n = cL$ where $\underbrace{c \in \mathbb{R}}_{\text{in}}$

(3) $\lim_{n \to \infty} a_n b_n = LK$

(4) $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{L}{K}$ where $b_n \neq 0$ and $K_n \neq 0$

Examples: $\{a_n\}$, $a_n = \frac{n}{n+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1}{\underset{\substack{\downarrow \\ 1}}{1} + \underset{\substack{\downarrow \\ 0}}{1/n}} = 1/1 = 1$$

<u>Thm</u> Let $f : \mathbb{R} \to \mathbb{R}$ be a function

Let $a_n = f(n)$, $n = 1, 2, 3 \ldots$

IF $\lim_{x\to\infty} f(x) = L$

THEN $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = L$

Example:

$\{a_n\}$, $a_n = (1 + 2/n)^n$, $\lim_{n\to\infty} a_n = ?$

Let $f(x) = (1 + 2/x)^x$. Then $a_n = f(n)$

Let us compute $\lim_{x\to\infty} f(x)$

$$\lim_{x\to\infty} (1 + 2/x)^x$$

$$= e^{\lim_{x\to\infty} \ln((1+2/x)^x)}$$

$$\lim_{x\to\infty} \ln((1+2/x)^x) = \lim_{x\to\infty} x \ln(1 + 2/x)$$

$$= \lim_{x\to\infty} \frac{\ln(1 + 2/x)}{1/x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+2/x} \cdot \left(\frac{-2}{x^2}\right)}{-1/x^2}$$

$$= \lim_{x\to\infty} \frac{2}{1 + 2/x} = 2$$

$$= e^2$$

<u>Thm</u> (Squeeze Thm)

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences

such that: $\lim a_n \Rightarrow \lim c_n \Rightarrow L$

and there is $N \in \mathbb{N}$ such that

$$a_n \le b_n \le c_n \quad \text{for } n > N$$

Then: $\lim b_n = L$

Examples

① $\{b_n\}$, $b_n = \frac{1}{n}\sin^2(n)$

Let $a_n = 0$

$c_n = 1/n \qquad n = 1, 2, \ldots$

Then $a_n \le b_n \le c_n$ and $\lim a_n = \lim c_n = 0$

Hence by the squeeze thm $\lim b_n = 0$

Thm (Absolute Value thm)

If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

Proof: $\quad -|a_n| \le a_n \le a_n$ for all n

$\quad 0 \qquad\qquad 0$

Then by the squeeze thm. $\lim a_n = 0$

Example: $\{a_n\}$, $a_n = \frac{(-1)^n}{n}$

$a_1 = -1$

$a_2 = 1/2$

$a_3 = -1/3$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

Hence, $\lim \frac{(-1)^n}{n} = 0$

$\{a_n\}$, $a_n = n/(n+1)$

$a_n \boxed{\le} a_{n+1}$

$2n/(n+1) \boxed{\le} 2n+1/(n+1+1)$

$\frac{2n}{n+1} \boxed{\le} \frac{2n+2}{n+2}$

$2n(n+2) \boxed{\le} (2n+2)(n+1)$

$2n^2 + 4n \boxed{\le} 2n^2 + 4n + 2$

$0 \boxed{\le} 2$

Defn (Monotonic Seq.)

A sequence $\{a_n\}$ is monotonic when its term are nondecreasing

$a_1 \le a_2 \le a_3 \le \ldots$

or when nonincreasing

$a_1 \ge a_2 \ge a_3 \ge \ldots$

$\{a_n\}$ is a non-decreasing

is monotonic.

<u>Def'n</u> (Bounded seqs.)

Let $\{a_n\}$ be a seq.

1. We say $\{a_n\}$ is <u>bounded above</u> if there is M such that $a_n \le M$ for all n.
2. We say $\{a_n\}$ is <u>bounded below</u> —— —— M such that $M \le a_n$ for all n
3. —— $\{a_n\}$ —— <u>bounded</u> —— —— M such that $-M \le a_n \le M$

$$\Updownarrow$$

$$(|a_n| \le M)$$

<u>Prop.</u> $\{a_n\}$ is bounded if and only if $\{a_n\}$ is bounded above and below.

<u>Examples</u>

1. $a_n = \frac{1}{n} \sin^2(n)$

$a_n \le 1/n \le 1 \Rightarrow \{a_n\}$ is bounded above

$0 \le a_n \Rightarrow$ —— is bounded below

So, $\{a_n\}$ is bounded.

Lecture: Sequences (Section 9.1)
Series and Convergence (Section 9.2)

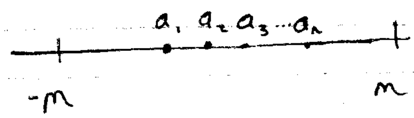
$\{a_n\}$ is non-decreasing if
$a_1 \le a_2 \le a_3 \le \dots$

$\{a_n\}$ is non-increasing if
$a_1 \ge a_2 \ge a_3 \ge \dots$

$\{a_n\}$ is monotonic if it is either non-increasing or non-decreasing

$\{a_n\}$ is bounded if there is $M > 0$ such that
$-M \le a_n \le M$ for all n

<u>Thm</u> (Convergence theory)
If a seq. is monotonic and bounded then it is convergent.



Example:
$\{a_n\}$,
$a_{n+1} = a_n^2 - a_n + 1$
$a_1 = 1/2$
$a_2 = a_{1+1} = a_1^2 - a_1 + 1$
$= 1/2^2 - 1/2 + 1$
$= 3/4$

Monotonic:
$a_{n+1} - a_n = a_n^2 - a_n + 1 - a_n$
$= a_n^2 - 2a_n + 1$
$= (a_n - 1)^2 \ge 0$

$\Rightarrow a_{n+1} \ge a_n$ for all n
$\Rightarrow a_1 \le a_2 \le a_3 \le \dots$
$\Rightarrow \{a_n\}$ is nondecreasing so it is monotonic

$a_n \ge a_{n-1} \ge \dots \ge a_3 \ge a_2 \ge a_1 = 1/2 > 0$
So, $a_n > 0$ for all n
$\Rightarrow \{a_n\}$ is bounded below by 0

$a_1 = 1/2$
$1 - a_2 = a_1(1 - a_1) \ge 0$
$1 - a_3 = a_2(1 - a_2) \ge 0$
$1 - a_4 = a_3(1 - a_3) \ge 0$
$\vdots$

$\to$ $1 - a_n \ge 0$ for all n $\Rightarrow a_n \le 1$ for all n
$\Rightarrow \{a_n\}$ is bounded above by 1

$\Rightarrow \{a_n\}$ is bounded

Hence, by the theorem $\{a_n\}$ converges

$\lim_{n\to\infty} a_n = L$ exists

then $\lim_{n\to\infty} a_{n+1} = L$ so

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (a_n^2 - a_{n+1}) = L^2 - L + 1$$

$$L = 1$$

<u>Series</u>

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = a_1 + \dots + a_n$$

<u>Def'n</u>: Convergence and divergence of Series

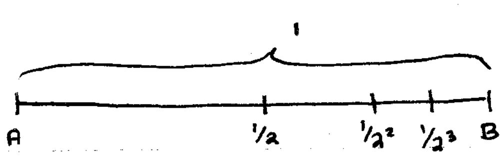
$S_n = a_1 + a_2 + \dots + a_n$ (n^{th} partial sum)

$\sum_{n=1}^{\infty} a_n$ converges if $\{S_n\}$ converges.

In which case we write $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$

$\sum_{n=1}^{\infty} a_n$ <u>diverges</u> if $\{S_n\}$ diverges

Example:



$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \dots$$

$$S_n = 1/2$$
$$S_2 = 1/2 + 1/2^2$$
$$S_3 = 1/2 + 1/2^2 + 1/2^3$$
$$\vdots$$
$$S_n = 1/2 + 1/2^2 + 1/2^3 + \dots + 1/2^n$$

$$\tfrac{1}{2}\, S_n = (1 - \tfrac{1}{2}) S_n = S_n - \tfrac{1}{2} S_n = \tfrac{1}{2} + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \dots + \tfrac{1}{2^n} - \tfrac{1}{2^2} - \tfrac{1}{2^3} - \dots - \tfrac{1}{2^n} - \tfrac{1}{2^{n+1}}$$

$$= \tfrac{1}{2} - \tfrac{1}{2^{n+1}}$$

$$S_n = 2\left(\tfrac{1}{2} - \tfrac{1}{2^{n+1}}\right) = 1 - \tfrac{1}{2^n}$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \tfrac{1}{2^n}\right) = 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \text{ converges}$$

(2) $\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 \dots$

$S_1 = 1$

$S_2 = 1 + 1 = 2$

$S_3 = 1 + 1 + 1 = 3$

$\vdots$

$S_n = \underbrace{1 + 1 + 1 + \dots}_{n} = n$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$ (divergent - not a number)

$\Rightarrow \sum_{n=1}^{\infty} 1$ diverges

(3) $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots \frac{1}{n(n+1)} + \dots$$

$S_1 = \frac{1}{1\cdot 2} = \frac{1}{1} - \frac{1}{2} = 1 - \tfrac{1}{2}$

$S_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = 1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3} = 1 - 1/3$

$\vdots$

$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} \dots \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$

then $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$

$\rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ converges

$$\frac{1}{n(n+1)} = \frac{n+1-n}{n(n+1)} = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)}$$
$$= \frac{1}{n} - \frac{1}{n+1}$$

Def'n: (Geometric sequence and series)

$\{a_n\}$, $a_n = ar^n$, $a \neq 0$, $r \in \mathbb{R}$ Geometric seq.

$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} ar^n$ Geometric Series

Example:
$a = 1$, $r = 1/2$, then $\sum_{n=0}^{\infty} \frac{1}{2}^n$

Thm: (convergence of a Geometric Series)

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \text{ (converges)} \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

Proof
$S_n = a + ar + ar^2 + \dots + ar^n$

$\Rightarrow (1-r)S_n = S_n - rS_n = a + ar + ar^2 + \dots ar^n$
$- ar - ar^2 \dots - ar^n - ar^{n+1}$
$= a - ar^{n+1}$

If $r \neq 1$, we get

$$S_n = a \frac{1-r^{n+1}}{1-r}$$

So, $\lim_{n\to\infty} S_n = \begin{cases} \frac{a}{1-r} & |r| < 1 \\ \text{diverges} \end{cases}$

Because $\lim_{n\to\infty} r^{n+1} = \begin{cases} 0 & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| < 1 \end{cases}$

Examples :

1. $$\sum_{n=0}^{\infty} \frac{5}{3^n} = \sum_{n=0}^{\infty} 5\left(\frac{1}{3}\right)^n = \frac{5}{1-1/3} = \frac{15}{2}$$

$a = 5$

$r = 1/3 \Rightarrow |r| = 1/3 < 1$

2. $$\sum_{n=0}^{\infty} \frac{3^{n+2}}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{3^2}{2} \cdot \frac{3^n}{2^n} = \sum_{n=0}^{\infty} \frac{9}{2}\left(\frac{3}{2}\right)^n$$

$= a \ \frac{9}{2}$

$= r = 3/2$

$\Rightarrow |r| > 1$

$\hookrightarrow$ diverges