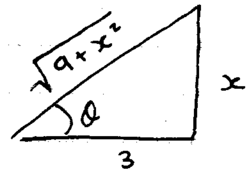
Lecture
- Trigonometric substitution (section 8.4)
- Partial fractions (section 8.5)

$\tan\theta = x/3$

$\sin\theta = x/(\sqrt{9+x^2})$

$\cos\theta = 3/(\sqrt{9+x^2})$

$\sec\theta = (\sqrt{9+x^2})/3$



For integrals involving $\quad \left(\begin{array}{l} a > 0 \\ u = u(x) \end{array} \right)$

1. $\sqrt{a^2-u^2}$

   let $u = a\sin\theta \quad (1-\sin^2\theta = \cos^2\theta)$

2. $\sqrt{a^2+u^2}$

   let $u = a\tan\theta$

3. $\sqrt{u^2-a^2}$

   let $u = a\sec\theta$

$$\left(\begin{array}{l} 1+\tan^2\theta = \sec^2\theta \\ \sec^2\theta - 1 = \tan^2\theta \end{array} \right)$$

Examples

1. $$\int \frac{dx}{x^2\sqrt{9-x^2}} \Rightarrow \int \frac{3\cos\theta}{9\sin^2\theta\sqrt{9-9\sin^2\theta}}\,d\theta$$

$\sqrt{9-x^2}$

$a = 3$

$u = x$

$x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$$= \int \frac{3\cos\theta}{9\sin^2\theta\sqrt{9\cos^2\theta}}\,d\theta$$

$$= \int \frac{3\cos\theta}{9\sin^2\theta \cdot 3\cos\theta}\,d\theta$$

$$= \frac{1}{9}\int \frac{1}{\sin^2\theta}\,d\theta = -\frac{1}{9}\cot\theta + C$$

$$\cot\theta = \frac{\sqrt{9-x^2}}{x}$$

2. $$\int \frac{dx}{\sqrt{4x^2+1}} = \int \frac{\frac{1}{2}\sec^2\theta}{\sqrt{\tan^2\theta+1}}\,d\theta = \int \frac{\frac{1}{2}\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta$$

$a = 1$

$u = 2x$

$2x = \tan\theta \Rightarrow x = \frac{1}{2}\tan\theta$

$\Rightarrow dx = \frac{1}{2}\sec^2\theta\,d\theta$

$$= \int \frac{\frac{1}{2}\sec^2\theta}{\sec\theta}\,d\theta \quad (\text{IF } \sec\theta \geq 0)$$

$$= \int \frac{1}{2}\sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$\tan\theta = 2x$

$\sec\theta = \sqrt{4x^2+1}$

3 $\displaystyle\int \frac{dx}{(x^2+1)^{3/2}} \Rightarrow \int \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\,d\theta$ IF $\sec\theta > 0$

$a = 1$
$u = x$
$x = \tan\theta$
$\Rightarrow dx = \sec^2\theta\,d\theta$

$\displaystyle\Rightarrow \int \frac{\sec^2\theta}{(\sec\theta)^3}\,d\theta$

$\displaystyle\Rightarrow \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\,d\theta$

$= \sin\theta + C$

$\displaystyle= \frac{x}{\sqrt{1+x^2}} + C$

$\left(\begin{array}{l} \tan\theta = x) \\ \sin\theta = x/\sqrt{1-x^2} \end{array}\right)$

Partial Fractions

Method to compute integrals of the form

$\displaystyle\int \frac{P(x)}{Q(x)}\,dx$ where P and Q are polynomials

Example : $\displaystyle\int \frac{x^3+3x+1}{x^2+x+1}\,dx$

1 IF degree of P ≥ degree of Q, then use long division.

$\displaystyle\int \frac{x^3}{x^2-1}\,dx$ $\left(\begin{array}{l} \deg(x^3) = 3 \\ \deg(x^2-1) = 2 \end{array}\right.$

$$\begin{array}{r} x \\ x^2-1 \overline{)\, x^3 \qquad} \\ -(x^3 - x) \\ \hline x \end{array}$$

$\displaystyle\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$

$$\begin{array}{r} T(x) \\ Q(x) \overline{)\, P(x)} \\ \downarrow \\ R(x) \end{array}$$

$\displaystyle\therefore \frac{P(x)}{Q(x)} = T(x) + \frac{R(x)}{Q(x)}$

$\displaystyle= \int\left(x + \frac{x}{x^2-1}\right)dx$

$\displaystyle= \frac{x^2}{2} + \int \frac{x}{x^2-1}\,dx$

2 a) Decompose $Q(x)$ in factors of the form
$(x-d)^n$ and $(x^2+ax+b)^m$

b) For each factor of the form
$(x-d)^n$ consider $\frac{D_1}{x-d} + \frac{D_2}{(x-d)^2} + \dots \frac{D_n}{(x-d)^n}$

$(x^2+ax+b)^m$ consider
$$\frac{A_1x+B_1}{x^2+ax+b)} + \frac{A_2x+B_2}{(x^2+ax+b)^2} + \dots + \frac{A_mx+B_m}{(x^2+ax+b)^m}$$

Then $\frac{P(x)}{Q(x)}$ is the sum of all those terms

c) compute $D_1, D_2, \dots, D_n$
$A_1, B_1, \dots, A_m, B_m$

d) compute the integrals

$$\int \frac{x^3}{x^2-1}\,dx = \frac{x^2}{2} + \int \frac{x}{x^2-1}\,dx = \frac{x^2}{2} + \int\left(\frac{1/2}{x+1} + \frac{1/2}{x-1}\right)dx$$
$$= \frac{x^2}{2} + 1/2 \ln|x+1| + 1/2 \ln|x-1| + C$$

$x^2-1 = (x+1)(x-1)$

$x+1 \rightsquigarrow \frac{A}{x+1}$

$x-1 \rightsquigarrow \frac{B}{x-1}$

$$\int \frac{1}{x(x^2+1)}\,dx$$

$\deg(1) = 0$

$\deg(x(x^2+1)) = 3$

$x \rightsquigarrow \dfrac{A}{x}$

$x^2+1 \rightsquigarrow \dfrac{Cx+D}{x^2+1}$

So, $\dfrac{1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Cx+D}{x^2+1}$

$\Rightarrow 1 = A(x^2+1) + (Cx+D)x$

$= (A+C)x^2 + Dx + A$

$\Rightarrow A + C = 0, \Rightarrow C = -1$

$D = 0$

$A = 1$

$$\frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$$

$$\int \frac{1}{x(x^2+1)}\,dx = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx$$

$$= \ln|x| - \tfrac{1}{2}\ln(x^2+1) + c$$

Integration by parts (Section 8.2)
Trigonometric Integrals (- 8.3)

NOTE:
$$\int u v'\, dx = uv - \int u' v\, dx$$

(1)
$$\int \frac{\ln x}{x^4}\, dx = -x^{-3} \ln x - \int (-1/3\, x^{-4})\, dx$$
$$= \frac{-x^{-3}}{3} \ln x - \frac{x^{-3}}{9} + C$$

$u = \ln x \Rightarrow u' = 1/x$
$v' = x^{-4} \Rightarrow v = x^{-3}/-3$

(2)
$$\int_0^{\pi/4} x^2 \cos x\, dx = x^2 \sin x \Big|_0^{\pi/4} - \int_0^{\pi/4} 2x \sin x\, dx = \frac{\pi^2}{16} \cdot \frac{\sqrt{2}}{2} - 2\int_0^{\pi/4} x \sin x\, dx$$

$u = x^2 \Rightarrow u' = 2x$
$v' = \cos x \Rightarrow v = \sin x$

$$\int_0^{\pi/4} x \sin x\, dx = -x\cos x \Big|_0^{\pi/4} - \int_0^{\pi/4} -\cos x\, dx$$
$$= -\frac{\pi}{4} \cdot \frac{\sqrt{2}}{2} + \int_0^{\pi/2} \cos x\, dx$$
$$= -\frac{\sqrt{2}}{8}\pi + \sin x \Big|_0^{\pi/4}$$
$$= \frac{-\sqrt{2}}{8}\pi + \frac{\sqrt{2}}{2}$$
$$\int_0^{\pi/4} x^2 \cos x\, dx = \frac{\sqrt{2}}{32}\pi^2 + \frac{\sqrt{2}}{4}\pi - \sqrt{2}$$

(3)
$$\int e^{2x} \sin x\, dx \Rightarrow 1/2\, e^{2x} \cdot \sin x - 1/2 \int e^{2x} \cos x$$

$u = \sin x \Rightarrow u' = \cos x$
$v' = e^{2x} \Rightarrow v = 1/2\, e^{2x}$

$u = \cos x \Rightarrow u' = -\sin x$
$v' = e^{2x} \Rightarrow v = 1/2\, e^{2x}$

$$\int e^{2x} \cos x\, dx = \frac{1}{2} e^{2x} \cos x - 1/2 \int -\sin x\, e^{2x}\, dx$$
$$= 1/2\, e^{2x} \cos x + 1/2 \int e^{2x} \sin x\, dx$$
(Δ)

$u = \cos x \Rightarrow u' = -\sin x$
$v' = e^{2x} \Rightarrow v = 1/2\, e^{2x}$

By $(*)$ and (Δ) ...

4 $$\int \frac{x^3 e^{x^2}}{(x^2+1)^2}\,dx = -\frac{x^2 e^{x^2}}{2(x^2+1)} - \int 2xe^{x^2}(x^2+1)\left(-\frac{1}{2(x^2+1)}\right)dx$$

$$u = x^2 e^{x^2} \Rightarrow 2xe^{x^2} + x^2(2x)e^{x^2} \Rightarrow 2xe^{x^2}(x^2+1)$$

$$v' = \frac{x}{(x^2+1)^2} \Rightarrow v = \int \frac{x}{(x^2+1)^2}dx \Rightarrow -\frac{1}{2(x^2+1)}$$

$$t = x^2+1$$
$$dt = 2x\,dx$$
$$dx = \frac{dt}{2x}$$

$$= \frac{-x^2e^{x^2}}{2(x^2+1)} + \int xe^{x^2}dx$$

$$\left(\begin{array}{l} t = x^2 \\ dt = 2x\,dx \\ = \int \tfrac{1}{2} e^t\,dt = \tfrac{1}{2}e^t = \tfrac{1}{2}e^{x^2} + c \end{array}\right)$$

$$= -\frac{x^2e^{x^2}}{2(x^2+1)} + \tfrac{1}{2}e^{x^2} + c$$

5 $$\int \sin^3(3x)\,dx = \int \sin^2(3x)\sin(3x)\,dx$$

$$= \int (1-\cos^2(3x))\sin(3x)\,dx$$

$$t = \cos(3x) \Rightarrow dt = -\sin(3x)\,dx$$

$$= \int (1-t^2)(-1/3)\,dt = -1/3\,(t - t^3/3) + C$$

$$= -\frac{\cos(3x)}{3} + \frac{\cos^3(3x)}{9} + C$$

(6) $\int_0^{\pi/2} \sin^4 x \, dx$

Note:

$\sin^2 x = \dfrac{1 - \cos(2x)}{2}$

$\cos^2 x = \dfrac{1 + \cos(2x)}{2}$

$\sin^4 x = (\sin^2 x)^2 = \left(\dfrac{1 - \cos(2x)}{2}\right)^2$

$= \frac{1}{4}\left(1 - 2\cos(2x) + \cos^2(2x)\right)$

$= \frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\dfrac{1 + \cos(4x)}{2}\right)$

$= \frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{8} + \frac{1}{8}\cos(4x)$

$\int_0^{\pi/2} \sin^4 x \, dx = \int_0^{\pi/2} \left(\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right) dx$

$= \left(\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x)\right\Big|_0^{\pi/2}$

Lecture Partial Fractions (Section 8.5)
Ideterminate Forms and L'Hopital Rule (section 8.7)

Examples

1. $\int \frac{1}{x^2-3x+2}\,dx$

$x^2-3x+2 = (x-2)(x-1)$

$(x-2) \longrightarrow \frac{A}{(x-2)}$

$(x-1) \longrightarrow \frac{B}{(x-1)}$

Hence

$$\frac{1}{x^2-3x+2} = \frac{A}{x-2} + \frac{B}{x-1} \Big/ (x-2)(x-1)$$

$\Rightarrow 1 = A(x-1) + B(x-2)$

Expanded: $Ax - A + Bx - 2B$

$0x + 1 = (A+B)x - A - 2B$

$\Rightarrow A + B = 0$

$\Rightarrow -A - 2B = 1 \quad \Rightarrow -A - 2(-A) = 1$

$\Rightarrow A = 1 \quad (B = -A)$

$\Rightarrow B = -1$

So, $\frac{1}{x^2-3x+2} = \frac{1}{x-2} - \frac{1}{x-1}$

$$\Rightarrow \int \frac{1}{x^2-3x+2}\,dx = \int \frac{1}{x-2}\,dx - \int \frac{1}{x-1}\,dx$$

$$= \ln|x-2| - \ln|x-1| + C$$

2. $\int \frac{x^4-2x^3+2x^2+1}{x^3-2x^2+x}\,dx$

$\deg(x^4-2x^3+2x^2+1) = 4$
$\deg(x^3-2x^2+x) = 3$

$$\begin{array}{r|l} & x \\ x^3-2x^2+x & x^4-2x^3+2x^2+1 \\ & -(x^4-2x^3+x^2) \\ \hline & 0 \quad 0 + x^2 + 1 \end{array}$$

$$\frac{x^4-2x^3+2x^2+1}{x^3-2x^2+x} = x + \frac{x^2+1}{x^3-2x^2+x}$$

$$\int \_\_\_\_\_\_ \,dx = \int x\,dx + \int \frac{x^2+1}{x^3-2x^2+x}\,dx$$

$$= \frac{x^2}{2} + \_\_\_\_\_\_\_\_$$

$$x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x-1)^2$$

$$x \longrightarrow \frac{A}{x}$$

$$(x-1)^2 \longrightarrow \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Hence

$$\frac{x^2+1}{x^3-2x^2+x} = \frac{x^2+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \Big/ \quad x(x-1)^2$$

$$\begin{aligned} x^2+1 &= A(x-1)^2 + Bx(x-1) + Cx \\ &= A(x^2-2x+1) + B(x^2-x) + Cx \\ &= Ax^2 - 2Ax + A + Bx^2 - Bx + Cx \\ &= (A+B)x^2 + (-2A-B+C)x + A \end{aligned}$$

$$-2A - B + C = 0$$

$$A = 1$$

Hence

$A + B = 1$

$-2A - B + C = 0$

$A = 1$

$C = 2$

$$\frac{x^2+1}{x^3-2x^2+x} = \frac{1}{x} + \frac{2}{(x-1)^2}$$

$$\int \underline{\qquad} \, dx = \int \frac{1}{x}\, dx + \frac{2}{(x-1)^2} + 2\int \frac{1}{(x-1)^2}\, dx = \ln|x| - \frac{2}{x-1} + C$$

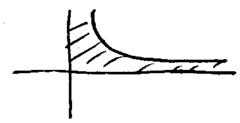
$$\int \frac{x^2+1}{x^3+3x-5}\,dx = \int \frac{1/3}{u}\,du = \frac{1}{3}\ln|u| + c$$

$$= \tfrac{1}{3}\ln|x^3+3x-5| + c$$

$u = x^3 + 3x - 5$

$du = (3x^2+3)\,dx$

$\quad = 3(x^2+3)\,dx$



Thm. (L'Hospital Rule)

IF:

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(c) = 0$$

Then:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

Let f and g be differentiable on the interval (a, b) and let c be in (a, b). Assume that $g'(x) \neq 0$ on (a, b) except possible at c.

Same holds if

$$\lim_{x\to c} f(x) \pm \infty$$

$$\lim_{x\to c} g(x) \pm \infty$$

Examples

derivative

(1) $$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1$$

$$\lim_{x\to 0} \sin x = \sin 0 = 0$$

$$\lim_{x\to 0} x = 0$$

(2) $$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}$$

$$\lim_{x\to 0} (1-\cos x) = 1 - \cos 0 = 0$$

$$\lim_{x\to 0} x^2 = 0$$

3 $\lim_{x \to \infty} \frac{\ln x}{x^2} = \lim_{x \to \infty} \frac{1/x}{2x} = \lim_{x \to \infty} \frac{1}{2x^2} = 0$

$\lim_{x \to \infty} \ln x = \infty$

$\lim_{x \to \infty} x^2 = \infty$

$\lim_{x \to c} \frac{f(x)}{g(x)}$ is not determinate $\left(\frac{0}{0}\right)$

$\lim_{x \to 0} \frac{x}{x} = 1$ $\qquad \lim_{x \to 0} \frac{2x}{x} = 2$

Lecture:
- indeterminate forms and l'hopital rule (8.2)
- Improper integrals (8.8)

L'Hopital Rule

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \underbrace{\lim_{x\to c} \frac{f'(x)}{g'(x)}}_{\text{If this limit exists or is equal to } \infty}$$

$$\left(\frac{0}{0}\right) \qquad \left(\frac{\pm\infty}{\pm\infty}\right)$$

$(0, \infty)$

$$\lim_{x\to c} \underset{\to 0}{f(x)}\,\underset{\to\infty}{g(x)} = \lim_{x\to c} \frac{f(x)}{1/g(x)} = \lim_{x\to c} \frac{g(x)}{1/f(x)}$$

Examples

1. $\lim_{x\to\infty} x^3 e^{-x} \Rightarrow \lim_{x\to\infty} \cancel{x^3}^{\,\infty}\ \cancel{e^{-x}}^{\,0} \qquad \left(\frac{1}{e^x}\right)$

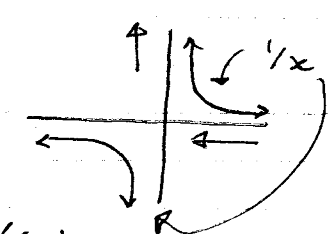
$$\Rightarrow \lim_{x\to\infty} \frac{x^3}{e^x} \Rightarrow \left(\frac{\infty}{\infty}\right)$$

$$\lim_{x\to\infty} \frac{3x^2}{e^x} \Rightarrow \left(\frac{\infty}{\infty}\right)$$

$$\lim_{x\to\infty} \frac{6x}{ex} \Rightarrow \left(\frac{\infty}{\infty}\right)$$

$$\lim_{x\to\infty} \frac{6}{e^x} \Rightarrow 0$$

2. $\lim_{x\to 0^+} (1+2x)^{1/x}$



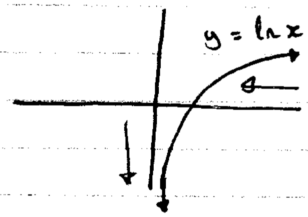
Step 1: $\lim_{x\to 0^+} \ln\left((1+2x)^{1/x}\right)$

$$= \lim_{x\to 0^+} \frac{\ln(1+2x)}{x} = \lim_{x\to 0^+}\left(\frac{0}{0}\right)$$

$$= \frac{\frac{1}{1+2x}\cdot 2}{1} = 2 \qquad 2$$

Step 2: $\lim\limits_{x \to 0^+} (1 + 2x)^{1/x} = e^2$

$\lim\limits_{x \to 0^+} (\tan x)^x$



Step 1: $\lim\limits_{x \to 0^+} \ln\left((\tan x)^x\right)$

$= \lim\limits_{x \to 0^+} \overset{0}{\cancel{x}}\left[\overset{-\infty}{\ln(\cancel{\tan} x)}\right]$

$= \lim\limits_{x \to 0^+} \dfrac{\ln[\tan x]}{1/x} \quad \left(\dfrac{-\infty}{\infty}\right)$

$= \lim\limits_{x \to 0^+} \dfrac{\frac{1}{\tan x} \cdot \sec^2 x}{-1/x^3}$

$\dfrac{\cos x}{\sin x} \cdot \dfrac{1}{\cos^2 x} \cdot x^2$

$= \lim\limits_{x \to 0^+} \dfrac{\cancel{\cos x}}{\sin x} \cdot \dfrac{1}{\cos^{\cancel{2}} x} \cdot x^2$

$= \lim\limits_{x \to 0^+} \dfrac{x}{\sin x} \cdot \dfrac{1}{\cos x} \cdot x$

$= 0$

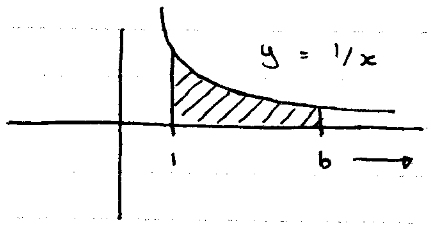
3 $\lim\limits_{x \to 1^+} \left(\overset{\infty}{\dfrac{1}{\ln x}} - \overset{\infty}{\dfrac{1}{x-1}} \right)$

$= \lim\limits_{x \to 1^+} \dfrac{x - 1 - \ln x}{(x-1)\ln x} = \lim\limits_{x \to 1^+} \dfrac{1 - 1/x}{1 \cdot \ln x + (x-1) \cdot 1/x} \cdot \dfrac{x}{x}$

$= \lim\limits_{x \to 1} \dfrac{x - 1}{x \ln x + x - 1} = \lim\limits_{x \to 1} \dfrac{1}{1 \cdot \ln x + x \cdot 1/x + 1}$

$(0/0)$

$= \lim\limits_{x \to 1} \dfrac{1}{\ln x + 2} = 1/2$



$$\int_1^\infty 1/x \, dx = \lim_{b \to \infty} \int_1^b 1/x \, dx$$

$$= \lim_{b \to \infty} \ln b = \infty$$

$$\int_1^\infty 1/x^2 \, dx = \lim_{b \to \infty} \int_1^b 1/x^2 \, dx = \lim_{b \to \infty} \left(-1/b + 1\right)$$

$$\int_1^b 1/x^2 \, dx = -\frac{1}{x}\Big|_1^b \Rightarrow -\frac{1}{b} + 1$$