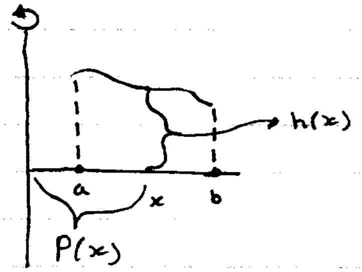
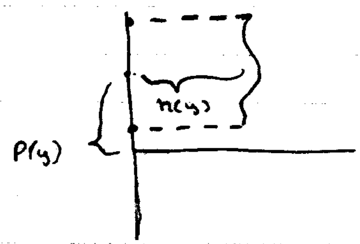
Shell Method



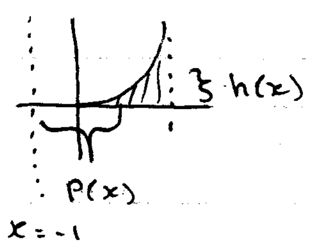


$$V = 2\pi\int_a^b P(x)h(x)\,dx \qquad\qquad V = 2\pi\int_c^d p(y)h(y)\,dy$$

Find the volume of the solid obtained by revolving the region bounded by

1. $y = x^3$, $y = 0$, $x = 1$ about $x = -1$

Sol:



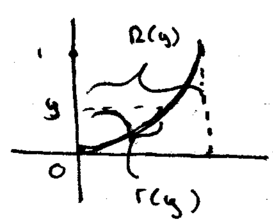
$P(x) = 1 + x$

$h(x) = x^3$

$$V = 2\pi\int_0^1 (1+x)x^3\,dx$$

$$= 2\pi\left(\frac{x^4}{5} + \frac{x^5}{5}\right)\Big|_0^1 = \frac{9}{10}\pi$$

<u>Washer method:</u>



$$V = \pi\int_c^d \left(R(y)^2 - r(y)^2\right)$$

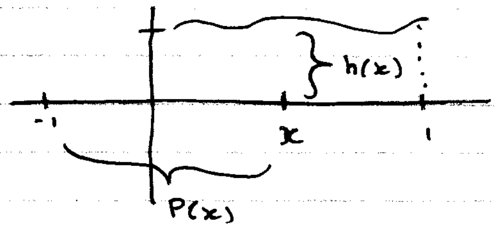
$r(y) = \sqrt[3]{y+1}$

$R(y) = 2$

$$V = \int_0^1 \left(2^2 - (\sqrt[3]{y} + 1)^2\right)dy$$

2) $y = x^7 + x^3 + 1$
bounded by $y = 0$, $x = 0$, $x = 1$, about $x = -1$

Solution



$$V = 2\pi \int_0^1 P(x)\,h(x)\,dx$$

$$V = 2\pi \int_0^1 (1+x)(x^7 + x^3 + 1)\,dx$$

$P(x) = 1 + x$

$h(x) = x^7 + x^3 + 1$

DISC METHOD
WASHER METHOD
SHELL METHOD
CROSS-SECTION METHOD

<u>Arc-length</u>

$y = f(x)$, $a \le x \le b$

$$S = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

$x = f(y)$ $\quad c \le y \le d$

$$S = \int_c^d \sqrt{1 + (f'(y))^2}\,dy$$

3) Find the arc-length of the curve
$y = \ln(\cos x)$, $0 \le x \le \pi/4$

<u>Solution</u>

$$S = \int_0^{\pi/4} \sqrt{1 + (f'(x))^2}\,dx$$

$f(x) = \ln(\cos x)$

$f'(x) = \dfrac{1}{\cos x}(-\sin x) \Rightarrow -\tan x$

$$S = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,dx$$

$$S = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx$$

$$S = \int_0^{\pi/4} \sec x\,dx$$

$= \ln|\sec x + \tan x| \Big|_0^{\pi/4}$

$= \ln|\sqrt{2} + 1| - \ln 1 \Rightarrow \ln|\sqrt{2} + 1| - 0$

Area of Surface of Revolution

→ $r(x)$ distance from the graph to the axis of revolution

$y = f(x)$, $a \le x \le b$

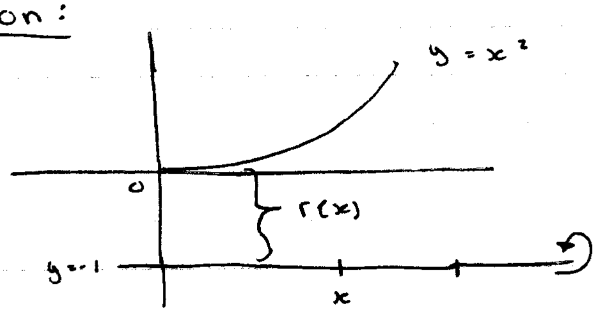
$$\text{Area} = 2\pi \int_a^b r(x) \sqrt{1 + (f'(x)^2)}\, dx$$

$x = f(y)$, $c \le y \le d$

$$\text{Area} = 2\pi \int_c^d r(y) \sqrt{1 + (f'(y)^2)}\, dy$$

$y = x^2$, $0 \le x \le 1$
about $y = -1$

Solution:



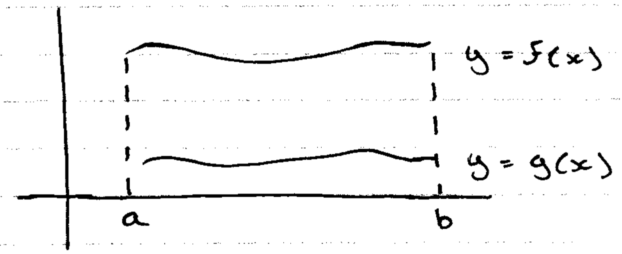
$$A = 2\pi \int_0^1 (x^2+1) \sqrt{1 + 4x^2}\, dx$$

$y = x^2$, $0 \le x \le 1$
about -2

Sol. $r(x) = x + 2$

$$A = 2\pi \int_0^1 (x+2) \sqrt{1 + 4x^2}\, dx$$

Mass, Moments, and Center of Mass



mass = $m = \rho \int_a^b (f(x) - g(x))\,dx$ = Density × Area

Moment about x-axis : $M_x = \rho \int_a^b \frac{f(x)^2 - g(x)^2}{2}\,dx$

y-axis : $M_y = \rho \int_a^b x\,(f(x) - g(x))\,dx$

Center of mass or centroid

$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$

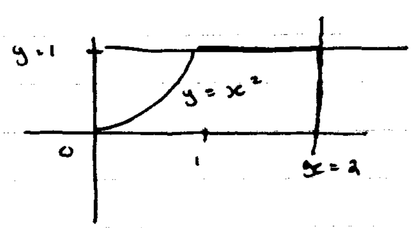
Find the mass of the planar lamina of density 2 given by the region bounded by

$y = x^2$, $x = 2$

$y = 0$

$y = 1$

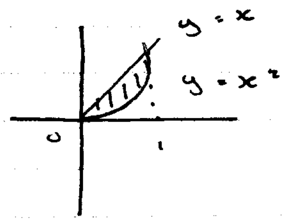
Solution:



$m = \rho \cdot \text{Area}$

$= 2\left(\int_0^1 x^2\,dx + \int_1^2 1\,dx \right)$

$y = x$
$y = x^2$



$$M = 2\int_0^1 (x - x^2)\,dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$M_x = 2\int_0^1 \frac{x^2 - x^4}{2}\,dx$$

$$= \left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{2}{15}$$

$$M_y = 2\int_0^1 x(x + x^2)\,dx = 2\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1$$
$$= 1/6$$

1 - Integrals → 4 - parts
↳ sub.
↳ inverse trig
↳ hyperbolic

1 - volumes, areas → 2 parts

1 - application of physics (work, etc.) hookes law.
↳ pumping water out of tanks

1 - arc - length or surface area

Hooke's Law

$F = kd$

F is Force required to compress or stretch a spring

d distance that the spring is compressed or stretched.

Examples: (Hookes Law)

A Force of 5 pounds compresses a 15 inch spring a total of 3 inches how much work done in compressing the spring 7 inches?

Solution:

$F(x) = kx$, $F(3) = 5$, $F(3) = k(3)$

So, $k = 5/3$

$\Rightarrow F(x) = \frac{5}{3}x$

$W = \int_0^7 F(x)\,dx \Rightarrow W = \int_0^7 5/3\,x\,dx$

$= 5/3 \frac{x^2}{2}\Big|_0^7 = \frac{245}{6}$ inch pound.

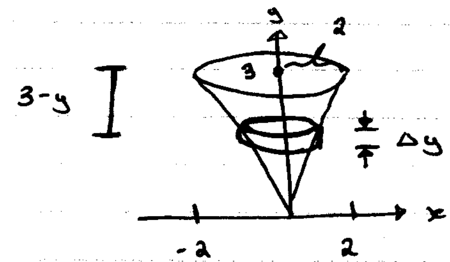
Example (Pumping water):

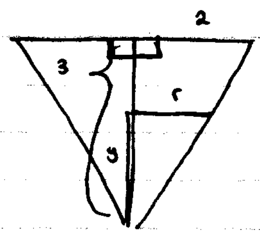
$$W = \int_a^b F(x)\,dx \longrightarrow \text{General Formula}$$

An open tank has a shape of right circular cone. The tank is 4-Feet across the top and 3-Feet high. How much work is done in emptying the tank by pumping the water over the top edge?

Sol: $\Delta W = \Delta F \cdot D$
$\quad = \Delta F\,(3-y)$

$\Delta F = \text{Density} \times \text{Volume}$
$\quad = 62.4 \times \pi r^2 \;\Delta y \;\to (\pi r^2 \cdot h)$





$$\frac{y}{3} = \frac{r}{2}$$

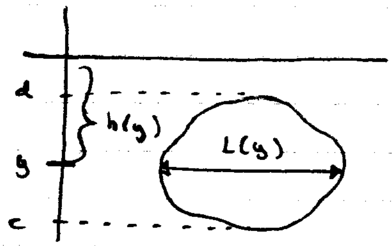
$$r = \frac{2y}{3}$$

$$\Delta W = 62.4 \times \pi\left(\frac{2y}{3}\right)^2 \Delta y$$

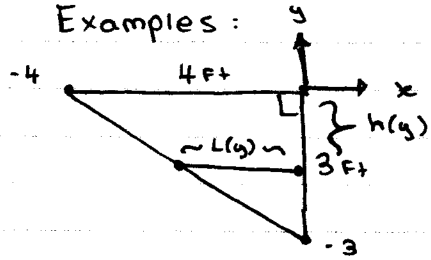
$$\Delta W = 62.4 \cdot \pi \cdot 4/9 \cdot y^2 \cdot (3-y)\,\Delta y$$

$$W = \int_0^3 62.4 \cdot \pi \cdot 4/9 \cdot y^2\,(3-y)\,dy$$

Fluid Force : $\rho \int_c^d h(y)\, L(y)\, dy$



Examples :



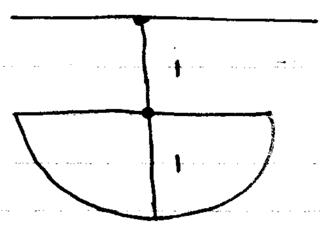
$h(y) = -y$
$L(y)$

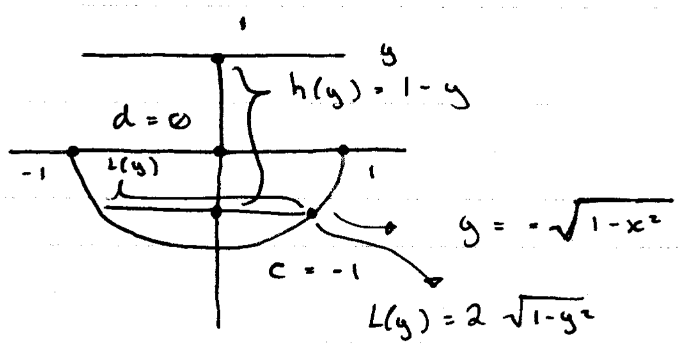
$\rho = 62.4$

$$\frac{L(y)}{4} = \frac{3+y}{3} \Rightarrow L(y) = \frac{4}{3}(3+y)$$

Fluid Force = $62.4 \int_{-3}^{0} (-y)\left(\tfrac{4}{3}(3+y)\right) dy$

Example :





$h(y) = 1 - y$
$L(y) = 2\sqrt{1-y^2}$

Fluid Force : $62.4 \int_{-1}^{0} (1-y)\left(2\sqrt{1-y^2}\right) dy$

Lecture Trigonometric Integrals (sec. 8.3, cont.)
Substitution (— 8.4)

$\int \sin^n x \cos^m x$

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = \dfrac{\cos(2x)+1}{2}$

$\sin^2 x = \dfrac{\sin(2x)+1}{2}$

$\int \sec^m x \tan^n x\,dx \rightarrow 1 + \tan^2 x = \sec^2 x$

$\int \csc^m x \cot^n x\,dx$

1 $\int \sec^{2k} x \tan^n x\,dx = \int \sec^{2k-2} x \tan^n x \sec^2 x\,dx$

$k > 1$

$= \int (\sec^2 x)^{k-1} \tan^n x \sec^2 x\,dx$

$= \int (1+\tan^2 x)^{k-1} \tan^n x \sec^2 x\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$= \int (1+u^2)^{k-1} u^n\,du$

Examples:

1 $\int \sec^4 x \tan^3 x\,dx = \int \sec^2 x \tan^3 x \sec^2 x\,dx$

$= \int (1+\tan^2 x)\tan^3 x \sec^2 x\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$= \int (1+u^2)u^3\,du = \int (u^3+u^5)\,du$

$= \dfrac{u^4}{4} + \dfrac{u^6}{6} + C$

$= \dfrac{\tan^4 x}{4} + \dfrac{\tan^6 x}{6} + C$

2 $\int \sec^m x \tan^{2k+1} x\,dx$

$m \geq 1$

$= \int \sec^{m-1} x \tan^{2k} x \cdot \sec x \tan x\,dx$

$= \int \sec^{m-1} x (\tan^2 x)^k \cdot \sec x \tan x\,dx$

$= \int \sec^{m-1} x (\sec^2 x - 1)^k \cdot \sec x \tan x\,dx$

$u = \sec x$

$du = \sec x \tan x\,dx$

$= \int u^{m-1}(u^2-1)^k\,du$

Examples:

① $\int \sec^3(2x)\tan^3(2x)\,dx$

$= \int \sec^2(2x)\tan^2(2x) \quad \sec(2x)\tan(2x)\,dx$

$= \int \sec^2(2x)(\sec^2(2x)-1) \quad \sec(2x)\tan(2x)\,dx$

$u = \sec(2x)$

$du = 2\sec(2x)\tan(2x)\,dx$

$= \int u^2(u^2-1)\,\frac{1}{2}du = \frac{1}{2}\int u^4 - u^2\,)\,du = \frac{u^5}{10} - \frac{u^3}{6} + C$

$= \frac{\sec^5(2x)}{10} - \frac{\sec^3(2x)}{6} + C$

③ $\int \tan^n x\,dx$

If $n = 1$, then $\int \tan x\,dx = -\ln|\cos x| + C$

If $n \geq 2$, then

$\int \tan^n x\,dx = \int \tan^{n-2}x\,\tan^2 x\,dx$

$= \int \tan^{n-2}x\,(\sec^2 x - 1)\,dx$

$= \int \tan^{n-2}x\,\sec^2 x\,dx - \int \tan^{n-2}x\,dx$

$u = \tan x$

$du = -\ln|\cos x|\,dx$

(repeat the same process)

Examples

① $\int \tan^5 x\,dx = \int \tan^3 x\,\tan^2 x\,dx$

$= \int \tan^3 x\,(\sec^2 - 1)\,dx \Rightarrow \int \tan^3 x\,\sec^2 x\,dx - \int \tan^3 x\,dx$

$\int \tan^3 x\,\sec^2 x\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{\tan^4 x}{4} + C$

$u = \tan x$

$du = \sec^2 x\,dx$

$\int \tan^3 x\,dx = \int \tan x\,\tan^2 x\,dx = \int \tan x\,(\sec^2 x - 1)\,dx$

$= \int \tan x\,\sec^2 x\,dx - \int \tan x\,dx = \frac{\tan^2 x}{2} - \ln|\cos x| + C$

$u = \tan x$

4 $\int \sec^n x \, dx$ $\qquad\qquad$ $\left(\int \csc^n x \, dx\right)$

Integration by parts

Example: $\boxed{\int \sec^3 x \, dx} = \sec x \tan x - \int \sec x \tan^2 x$

$u = \sec x \Rightarrow u' = \sec x \tan x$

$v' = \sec^2 x \qquad v = \tan x$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x \, dx$$

Let $A = \int \sec^3 x \, dx$ then

$$A = \sec x \tan x + \ln|\sec x + \tan x| - A$$

$$A = \tfrac{1}{2}\left(\_\_\_ + \_\_\_\right) + c = \boxed{\int \sec^3 x \, dx}$$

5 If none of the above, try changing to sines and cosines

Example: $\displaystyle\int \frac{\sec x}{\tan^2 x} \, dx \Rightarrow \int \frac{\frac{1}{\cos x}}{\frac{\sin^2 x}{\cos^2 x}} \, dx = \int \frac{\cos x}{\sin^2 x} \, dx$

$\sec x = \dfrac{1}{\cos x}$ $\qquad\qquad$ $= \int \cos x \sin^{-2} x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$\tan x = \dfrac{\sin x}{\cos x}$

$$= \int u^{-2} \, du = -u^{-1} + c$$

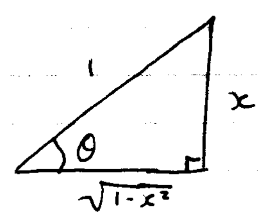
$$= -\frac{1}{\sin x} + c$$

$x = \sin \theta$

$\theta = \arcsin x$ or $(\sin^{-1} x)$

$\sin(2\theta) = 2 \sin\theta \cos\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$



$\hookrightarrow 1 - x^2 - x^2 = 1 - 2x^2$

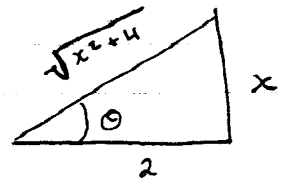
$\tan\theta = \dfrac{x}{\sqrt{1-x^2}}$

$\tan\theta = x/2$

$\sin\theta = x/\sqrt{x^2+4}$

$\cos\theta = 2/\sqrt{x^2+4}$

$\theta = \arctan(x/2)$



① Integrals involving $\sqrt{a^2-x^2}$

$x = a\sin\theta$

<u>Example</u> $\int \sqrt{1-x^2}\,dx \Rightarrow \int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$

$x = \sin\theta$

$dx = \cos\theta\,d\theta \Rightarrow \int \sqrt{\cos^2\theta}\,\cos\theta\,d\theta$

$= \int |\cos\theta|\cos\theta\,d\theta$

$-\pi/2 \le \theta \le \pi/2, \quad \cos\theta \ge 0$

$|\cos\theta| = \cos\theta$

$= \int \cos^2\theta\,d\theta$

$= \int \frac{\cos(2\theta)+1}{2}\,dx$

$= \frac{1}{2}\int \cos(2\theta)\,d\theta + \frac{1}{2}\int d\theta$

$\Rightarrow \frac{1}{4}\sin(2\theta) + \frac{\theta}{2} + C$

$= \frac{1}{4}\left(2x\sqrt{1-x^2}\right) + \frac{\arcsin x}{2} + C$