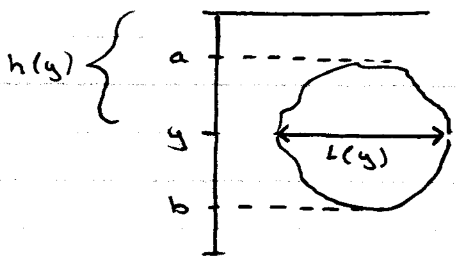
FEB. 6/17

Lecture 13 - Fluid Pressure and Fluid Force (sec. 7.2 cont)
Integration by parts (section 8.2)



Fluid Force

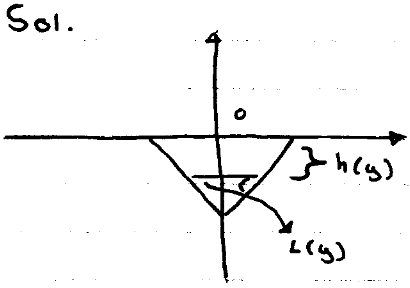
$$F = \rho \int_a^b L(y)\,h(y)\,dy$$

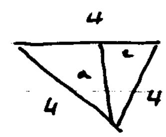
$L(y)$ horizontal length at y
$h(y)$ depth at y
ρ density of the fluid

Examples

1. Find the fluid force on the vertical side of a tank that is full of water if the table is an equilateral triangle of side 4 feet.

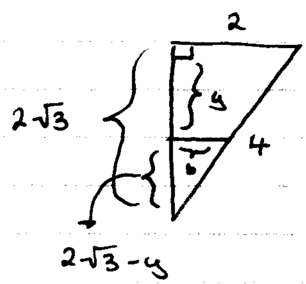
Sol.





2)

$a = \sqrt{4^2 - 2^2}$
$a = 2\sqrt{3}$



$h(y) = -y$
$L(y) = 2b$

$\left(2b = \dfrac{4\sqrt{3} - 2y}{\sqrt{3}} \right)$

$\rho = 62.4$

$\dfrac{b}{2} = \dfrac{2\sqrt{3} - y}{2\sqrt{3}}$

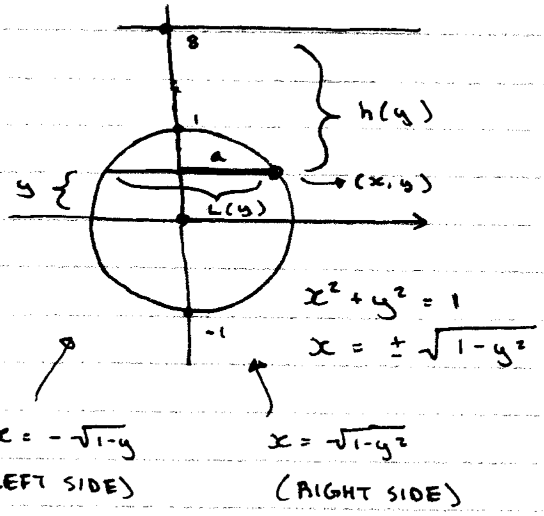
$\Rightarrow b = \dfrac{2\sqrt{3} - y}{\sqrt{3}}$

$$F = 62.4 \int_{-2\sqrt{3}}^{0} -y \left(\frac{4\sqrt{3} - 2y}{\sqrt{3}} \right) dy$$

$$F = \frac{62.4}{\sqrt{3}} \int_{-2\sqrt{3}}^{0} \left(-4\sqrt{3}\,y + 2y^2 \right) dy \quad \left[\text{therefore } \frac{62.4}{\sqrt{3}} \left(-2\sqrt{3}\,y^2 + \tfrac{2}{3} y^3 \right) \Big|_{-2\sqrt{3}}^{0} \right]$$

$\Rightarrow \simeq 499.2$ lbs

2 A circular observation window on a marine science ship has a radius of 1 foot and the center of the window is 8 feet below water level. What is the fluid force on the window?



$h(y) = 8 - y$

$L(y) = 2a = 2\sqrt{1-y^2}$

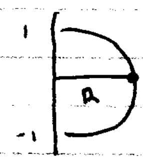
$F = 62.4\int_{-1}^{1}(8-y)(2\sqrt{1-y^2})\,dy$

$= 62.4\cdot 16\int_{-1}^{1}\sqrt{1-y^2} - 62.4\cdot 2\int_{-1}^{1} y\sqrt{1-y^2}\,dy$

$= 512\pi$ lbs

$t = 1 - y^2$

$\int_{-1}^{1}\sqrt{1-y^2}\,dy$ = Area of R $= \dfrac{\pi\cdot 1^2}{2} = \dfrac{\pi}{2}$



$x = \sqrt{1-y^2}$

Integration by parts:

$(uv)' = u'v + uv' \Leftrightarrow uv' = (uv)' - u'v$

$\Rightarrow \int uv'\,dx = \int (uv)'\,dx - \int u'v\,dx$

$= uv - \int u'v\,dx$

Thm (Integration by parts)

Let u and v be functions with continuous derivatives

Then $\int uv'\,dx = uv - \int u'v\,dx$

Sometimes written as:

$\left(\int u\,dv = uv - \int v\,du \qquad \begin{array}{l} dv = v'dx \\ du = u'dx \end{array}\right)$

Integrals that can be computed by using integration by parts : (a, b, constants)

① $\int x^n e^{ax}\,dx$, $\int x^n \sin(ax)\,dx$, $\int x^n \cos(ax)\,dx$

$u = x^n$, $v' = e^{ax}$, $\sin(ax)$, $\cos(ax)$

② $\int x^n \ln x\,dx$, $\int x^n \arcsin(ax)\,dx$, $\int x^n \arctan(ax)\,dx$

$u = \ln x$, $\arcsin(ax)$, $\arctan(ax)$, $v' = x^n$

③ $\int e^{ax} \sin(bx)$, $\int e^{ax} \cos(bx)\,dx$

$u = \sin(bx)$, $\cos(bx)$ $v' = e^{ax}$

Examples :

① $\int x e^{-x} = x(-e^{-x}) - \int 1\cdot(-e^{-x})\,dx$

$u = x \Rightarrow u' = 1$

$v' = e^{-x} \Rightarrow v = -e^{-x}$

$= -xe^{-x} + \int e^{-x}\,dx$

$= -xe^{-x} - e^{-x} + c$

$c \in \mathbb{R}$

② $\int x^2 \sin(2x)\,dx$

$u = x^2$ $\quad u = 2x$

$v' = \sin(2x)$ $\quad v = \int \sin(2x)\,dx = -\dfrac{\cos(2x)}{2}$

$$= -\frac{x^2}{2}\cos(2x) - \int 2x\left(-\frac{\cos(2x)}{2}\right)dx$$

$u = x$ $\quad u' = 1$

$v' = \cos(2x) \Rightarrow v = \dfrac{\sin(2x)}{2}$

$$= \frac{-x^2}{2}\cos(2x) + \int x\cos(2x)\,dx$$

$$= -\frac{x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) - \int \frac{\sin(2x)}{2}$$

$$= \frac{-x^2}{2}\cos(2x) + \frac{x}{2}\sin(2x) + \frac{\cos(2x)}{4} + c$$

$c \in \mathbb{R}$

LAB 5

Arc Length $\Rightarrow$ ① $y = f(x) \quad a \le x \le b$

$$S = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

② $x = f(y) \quad c \le y \le d$

$$S = \int_c^d \sqrt{1 + f(y)^2}\, dy$$

Examples

① $y = \cosh x \;,\; x \in [0, 2]$

$$S = \int_0^2 \sqrt{1 + \sinh^2 x}\, dx$$

Consider $1 + \sinh^2 x = \cosh^2 x$

$$S = \int_0^2 \sqrt{\cosh^2 x}\, dx = \int_0^2 \cosh x \, dx = \sinh x \Big|_0^2$$

$$= \sinh 2 - \sinh 0$$

$$= \frac{e^2 - e^{-2}}{2} - 0$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\sinh 2 = \frac{e^2 - e^{-2}}{2}$$

$$\sinh 0 = 0$$

② Show that $\int_0^1 \sqrt{1 + 4x^2 e^{-2x^2}}\, dx > \sqrt{2}$

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx \qquad a = 0, \quad b = 0$$

negative root

$$f'(x)^2 = 4x^2 e^{2x^2} \Rightarrow f'(x) = -2x e^{x^2}$$

$$\Rightarrow f(x) = \int -2x e^{-x^2}\, dx$$

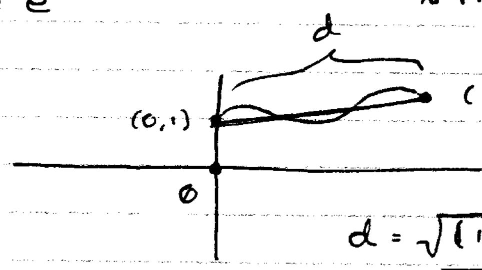
$$= e^{-x^2}$$

or set $t = x^2$ to find

$$f(x) = e^{x^2} \quad 0 \le x \le 1$$

$$f(0) = e^0 = 1$$

$$f(1) = e^1 = e$$



$$d = \sqrt{(1-0)^2 + (e-1)^2}$$

$$= \sqrt{1 + (e-1)^2}$$

So,

$$\int_0^1 \sqrt{1 + 4x^2 e^{2x^2}}\, dx > \sqrt{1 + (e-1)^2} > \sqrt{1+1} = \sqrt{2}$$

3 The arc length of the curve $y = f(x)$ from $(0,0)$ to $(x, f(x))$ is given by

$$s(x) = \int_0^x \sqrt{1 + e^t}\, dt$$

Find the equation of f.

Solution:

Arc length of $y = f(t)$ from $t = 0$ to $t = x$ is

$$s(x) = \int_0^x \sqrt{1 + f'(t)^2}\, dt = \int_0^x \sqrt{1 + e^t}\, dt$$

$\Rightarrow (f'(t))^2 = e^t \Rightarrow f'(t) = \sqrt{e^t} = (e^t)^{1/2} = e^{t/2}$

$f(t) = \int e^{t/2}\, dt \Rightarrow \int e^u\, 2du$

$\boxed{\begin{array}{l} \text{Let } u = t/2 \\ du = 1/2\, dt \\ dt = 2du \end{array}}$

$\Rightarrow 2e^u + C$

$\Rightarrow 2e^{t/2} + C$

$\Rightarrow f(t) = 2e^{t/2}$

$\Rightarrow f(x) = 2e^{x/2}$

Area of Surface of Revolution

1 $y = f(x) \qquad a \le x \le b$

$$S = 2\pi \int_a^b r(x) \sqrt{1 + (f'(x))^2}\, dx$$

$r(x)$ distance between the graph of f and the axis of revolution.

2 $x = f(y) \qquad c \le x \le d$

$$S = 2\pi \int_c^d r(y) \sqrt{1 + (f'(y))^2}\, dy$$

Examples

half a circle

1 $y = 1 + \sqrt{4 - x^2} \qquad -1 \le x \le 1 \quad$ about $y = 1$

consider

$y = \sqrt{4 - x^2}$

$\Rightarrow y^2 = 4 - x^2$

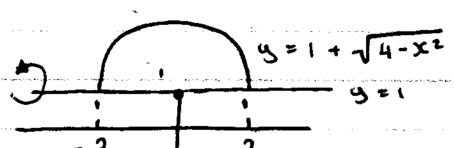
$\Rightarrow x^2 + y^2 = 4$

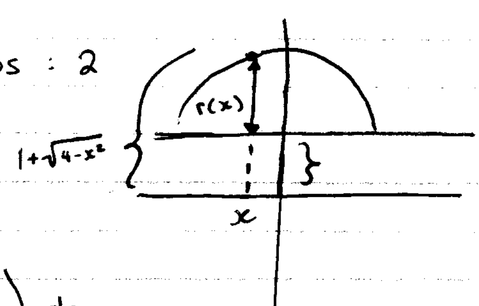
Solution:

$= (y-1)^2 = 4 - x^2$

$= (y-1)^2 + x^2 = 4$

Center: $(0,1)$, Radius: 2





$$S = 2\pi \int_{-1}^{1} (\sqrt{4 - x^2}) \left(\sqrt{1 + \frac{x^2}{4-x^2}} \right) dx$$

$$f(x) = 1 + \sqrt{4-x^2} = 1 + (4-x^2)^{1/2}$$
$$f'(x) = 1/2\,(4-x^2)^{-1/2} \cdot (-2x)$$

$$= 2\pi \int_{-1}^{1} \sqrt{4-x^2} \sqrt{\frac{4-x^2+x^2}{4-x^2}}\, dx$$

$$= 2\pi \int_{-1}^{1} 2\, dx$$

$$= 8\pi$$

Lecture · Integration by parts (section 8.2)
· Trigonometric Integrals (section 8.3)

$\int uv'\,dx = uv - \int u'v\,dx$ Integration by parts

a, b constants

① $\int x^n e^{ax}\,dx$, $\int x^n \sin(ax)\,dx$, $\int x^n \cos(ax)\,dx$

$u = x^n$

② $\int x^n \ln x\,dx$, $\int x^n \arcsin(ax)\,dx$, $\int x^n \arctan(ax)\,dx$

$u = \ln x$ $\quad u = \arcsin(ax)$ $\quad u = \arctan(ax)$

③ $\int e^{ax}\sin(bx)\,dx$, $\int e^{ax}\cos(bx)\,dx$

$u = \sin(bx)$ $\quad u = \cos(bx)$

$v' = e^{ax}$ $\quad v' = e^{ax}$

Examples

① $\int x^3 \ln x\,dx$

$u = \ln x \Rightarrow du = 1/x\,dx$

$v' = x^3 = \int x^3\,dx = \frac{x^4}{4}$

$\frac{1}{4}x^4 \ln x - \int \frac{1}{x}\left(\frac{x^4}{4}\right)dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$

$C \in \mathbb{R}$

② $\int \arcsin x\,dx$

$u = \arcsin x \Rightarrow u' = 1/\sqrt{1-x^2}$

$v' = 1 \Rightarrow \int 1\,dx = x$

$x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx \Rightarrow x\arcsin x + (1-x^2)^{1/2} + C$

$C \in \mathbb{R}$

Consider: $\int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{t}}\left(-\frac{1}{2}\right)dt = -\frac{1}{2}\int t^{-1/2}\,dt$

$t = 1 - x^2$

$dt = -2x\,dx$

$= \frac{-1/2\; t^{1/2}}{1/2} + C$

$= -(1-x^2)^{1/2} + C$

Examples:

(3) $\int x \arctan x \, dx$

$u = \arctan x \Rightarrow u' = 1/1+x^2$

$v' = x \Rightarrow \int v' dx = v = \frac{x^2}{2}$

$\frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$

consider: $\int \frac{x^2}{1+x^2} dx = \int \frac{1 + x^2 - 1}{1+x^2} dx$

$= \int \left(1 - \frac{1}{1+x^2}\right) dx$

$= x - \arctan x + C$

Then: $\frac{x^2}{2} \arctan x - \frac{1}{2} x + \arctan x + c, \quad C \in \mathbb{R}$

Examples:

(4) $\int e^x \sin x \, dx$

$u = \sin x \Rightarrow u' = \cos x$

$v' = e^x \Rightarrow \int v' dx = v = e^x$

$e^x \sin x - \int e^x \cos x \, dx$

$\Rightarrow e^x \sin x - \left(e^x \cos x - \int e^x (-\sin x) dx\right)$

$\Rightarrow e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x \, dx}_{\text{put } A = \int e^x \sin x \, dx}$

Then

$A = e^x \sin x - e^x \cos x - A$

$2A = e^x \sin x - e^x \cos x$

$A = \frac{1}{2}(e^x \sin x - e^x \cos x)$

$\Rightarrow \int e^x \sin x \, dx = \frac{1}{2}(e^x \sin x - e^x \cos x) + C$
$C \in \mathbb{R}$

TRIGONOMETRIC INTEGRALS

$\int \sin^n x \cos^m x \, dx$

(1) $\int \sin^{2k+1} x \cos^m x \, dx = \int (\sin^2 x)^k \cos^m x \sin x \, dx$

$= \int (1 - \cos^2 x)^k \cos^m x \sin x \, dx$

$= \int (1 - t^2)^k t^m (-1) \, dt$

(2) $\int \sin^n x \cos^{2n+1} x \, dx = \int \sin^n x \, (1-\sin^2 x)^n \cos x \, dx$

$t = \sin x$

(3) $\int \sin^{2n} x \cos^{2m} dx$

$\sin^2 x = \dfrac{1-\cos(2x)}{2}$

$\cos^2 x = \dfrac{1+\cos(2x)}{2}$

Example:

(1) $\int \sin^3 x \cos^2 x \, dx$

$= \int \sin^2 x \cos^2 x \sin x \, dx$

$= \int (1-\cos^2 x)\cos^2 x \sin x \, dx$

$t = \cos x$

$dt = -\sin x \, dx$

$= \int (1-t^2) t^2 (-1)\, dt$

$= \int (t^2 - t^4)\, dt = -1/3\, t^3 + 1/5\, t^5 + C$

$= -1/3 \cos^3 x + 1/5 \cos^5 x + C$

$C \in \mathbb{R}$

(2) $\int \cos^4 x \, dx = \int (\cos^2 x)^2 dx = \int \left(\dfrac{1+\cos(2x)}{2}\right)^2 dx$

$= 1/4 \int (1 + 2\cos(2x) + \cos^2(2x))\, dx$

$= 1/4 \int \left(1 + 2\cos(2x) + \dfrac{1+\cos(4x)}{2}\right) dx$

$= 1/4 \int \left(3/2 + 2\cos(2x) + 1/2 \cos(4x)\right) dx$

$= 3/8\, x + \sin(2x) + 1/8 \sin(4x) + C$

(3) $\int \sin^2 x \cos^2 x \, dx = \int 1/4 \sin^2(2x)\, dx$

$= 1/4 \int \dfrac{1-\cos(4x)}{2} dx$

$= 1/8\, x - 1/8 \sin(4x) + C$

$C \in \mathbb{R}$

$\sin^2 x = \dfrac{1-\cos(2x)}{2}$

$\cos^2 x = \dfrac{1+\cos(2x)}{2}$

$\sin(2x) = 2\sin x \cos x$

$$\int \frac{\cos^5 x}{\sqrt{\sin x}}\,dx = \int (\sin x)^{-1/2} \cos^5 x \, dx$$

$$= \int (\sin x)^{-1/2} \cos^4 x \cos x \, dx$$

$$= \int (\sin x)^{-1/2} (1 - \sin^2 x)^2 \cos x \, dx$$

$$t = \sin x \qquad dt = \cos x \, dx$$

$$= \int t^{-1/2} (1 - 2t^2 + 4t) \, dt$$

$$= \int (t^{-1/2} - 2t^{3/2} + t^{7/2}) \, dt$$

4-5 Questions

Feb.10|17

① Computation of Integrals

e.g. $\int \frac{(\ln x)^2}{x}\, dx$

Let $u = \ln x$

$du = 1/x\, dx$

$= \int u^2\, du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C \quad C \in \mathbb{R}$

For $\int_1^2 \frac{(\ln x)^2}{x}\, dx = \left(\frac{\ln^3 x}{3}\right)\Big|_1^2 = \frac{\ln 2^3}{3}$

$\int_1^2 \frac{(\ln x)^2}{x}\, dx = \int_0^{\ln 2} u^2\, du = \frac{u^3}{3}\Big|_0^{\ln 2} = \frac{\ln 2^3}{3}$

e.g. $\int \frac{1}{\sqrt{2-4x^2}}\, dx \xrightarrow{\text{note:}} \int \frac{u'}{\sqrt{a^2-u^2}}\, dx = \arcsin(u/a) + C$

$= \int \frac{1}{\sqrt{(\sqrt{2})^2-(2x)^2}}$ $\qquad a = \sqrt{2}$

$= \frac{1}{2}\int \frac{2}{\sqrt{(\sqrt{2})^2-(2x)^2}}\, dx$ $\qquad u = 2x \Rightarrow u' = 2$

$= \frac{1}{2}\arcsin\left(\frac{2x}{\sqrt{2}}\right) + C$

$\int \frac{1}{\sqrt{4x-x^2}}\, dx = \int \frac{1}{\sqrt{2^2-(x-2)^2}}\, dx = \arcsin\left(\frac{x-2}{2}\right) + C$

$4x - x^2 = -(x^2-4x) = -(x^2-4x+4-4)$

$= -((x-2)^2-4)$

$= 4-(x-2)^2$

Solve $\frac{dy}{dx} = \tan x \sec^2 x \Rightarrow y = \int \tan x \sec^2 x\, dx$

$y(0) = 2$

$u = \tan x$

$du = \sec^2 x\, dx$

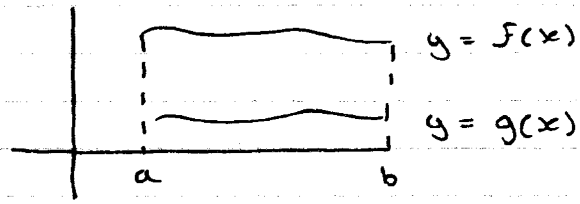
$y = \int u\, du = \frac{u^2}{2} + C$

$= \frac{\tan^2 x}{2} + C$

Hence

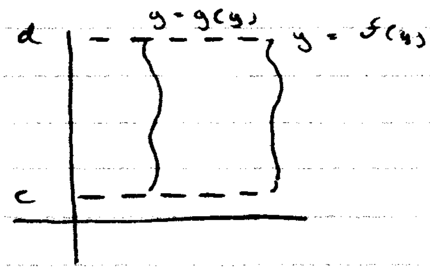
$y(0) = 2 = \frac{\tan^2(0)}{2} + C = C$

$y = \frac{\tan^2 x}{2} + 2$



1Q From work, Moments, and center of mass.

$$\text{Area} = \int_a^b (f(x) - g(x))\, dx$$

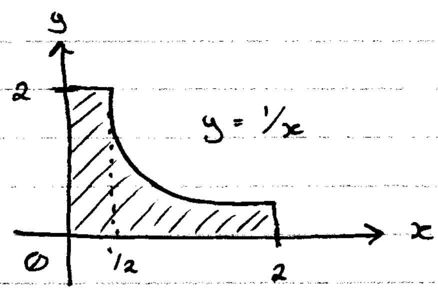


$$\text{Area} = \int_c^d (f(y) - g(y))\, dy$$

Compute the area of region bounded by :

$xy = 1$, $x = 2$, $y = 2$

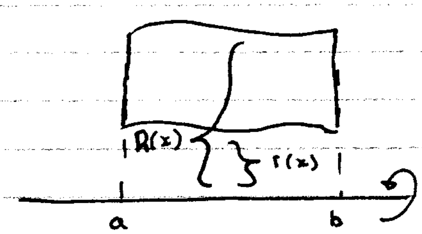
$x = 0$, $y = 0$



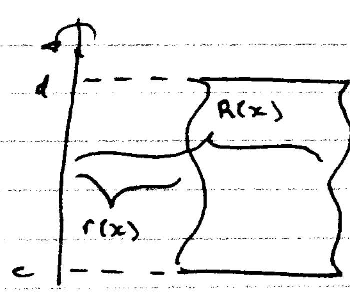
$$2 = 1/x \Rightarrow x = 1/2$$

$$\text{Area} = \int_0^{1/2} 2\,dx + \int_{1/2}^{2} \frac{1}{x}$$

$$= 2x \Big|_0^{1/2} + \ln x \Big|_{1/2}^{2}$$



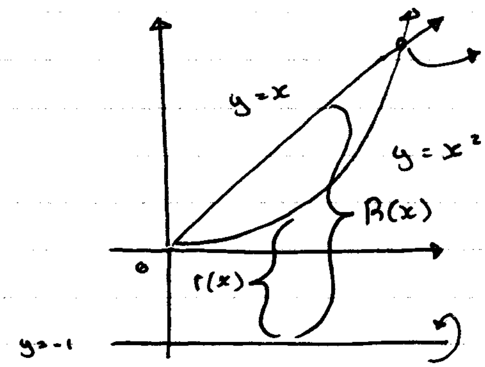
$$\text{Volume} = \pi \int_a^b (R(x)^2 - r(x)^2)\, dx$$



$$\text{Volume} = \pi \int_c^d (R(y)^2 - r(y)^2)\, dy$$

$y = x$

$y = x^2$ about the line $y = -1$

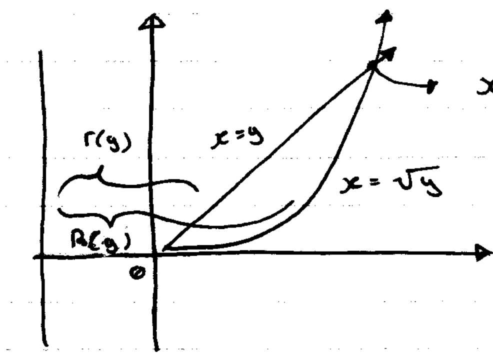


$x = x^2$ when $x = 0$, $x = 1$

$r(x) = 1 + x^2$

$R(x) = 1 + x$

$$\text{Volume} = \pi \int_0^1 \left[(1+x^2)^2 - (1+x)^2 \right] dx$$



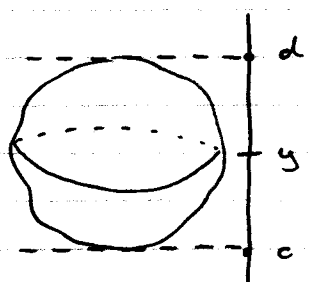
$x = x^2$ when $x = 0$, $x = 1$

$r(y) = 1 + y$

$R(y) = 1 + \sqrt{y}$

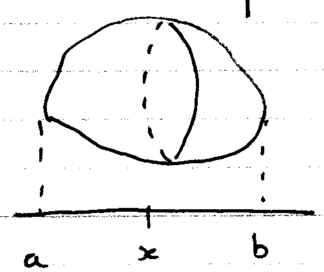
$$\text{Volume} = \pi \int_0^1 \left((\sqrt{y}+1)^2 - (1+y)^2 \right) dy$$

Crossed section method:



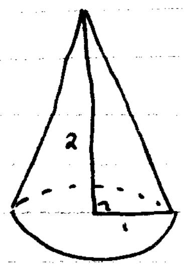
Area = $A(y)$

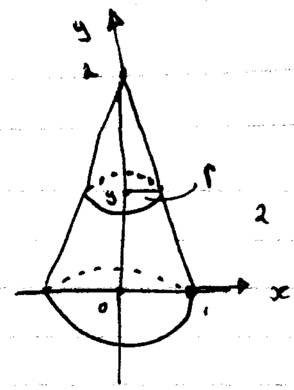
$$\text{Volume} = \int_c^d A(y)\,dy$$

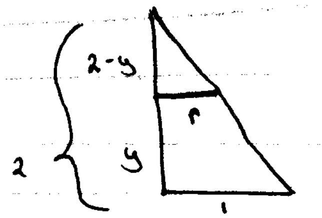


Area = $A(x)$

$$\text{Volume} = \int_a^b A(x)\,dx$$







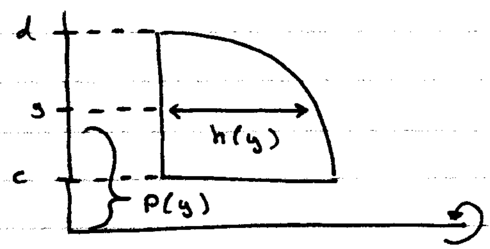
$$\frac{2-y}{2} = \frac{r}{1}$$

$$r = \frac{2-y}{2}$$

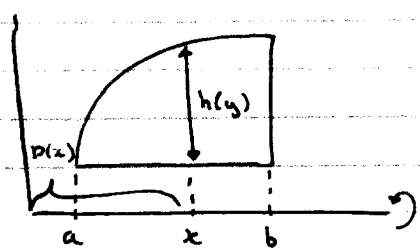
$$A(y) = \pi\left(\frac{2-y}{2}\right)^2$$

$$V = \int_0^2 \pi\left(\frac{2-y}{2}\right)^2 dy$$

Shell Method:



$$\text{Volume} = 2\pi\int_c^d P(y)\,h(y)\,dy$$



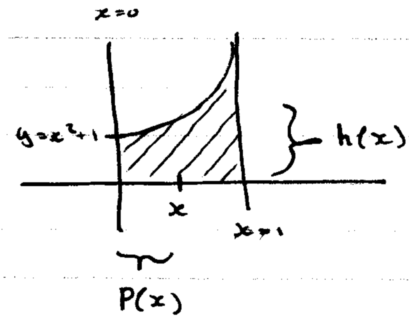
$$\text{Volume} = 2\pi\int_a^b P(x)\,h(x)$$

$y = x^2 + 1$

$y = 0 \quad , \quad x = 0$

$x = 1$

About y-axis



$P(x) = x$

$h(x) = x^2 + 1$

$$V = 2\pi \int_0^1 x\,(x^2+1)\,dx$$