Lecture 10 • Area of surface of revolution
• work (section 7.5)

Area of a surface of revolution

Let $y = f(x)$ represents a smooth curve on $[a, b]$. Then the area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x) \sqrt{1 + (f'(x))^2}\, dx$$

where $r(x)$ is the distance from the graph of f to the axis of revolution.

If $x = g(y)$ on $[c, d]$, then the surface area is

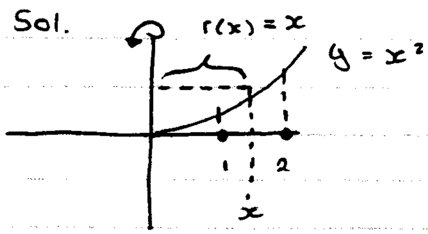
$$S = 2\pi \int_c^d r(y) \sqrt{1 + (g'(y))^2}\, dy$$

where $r(y)$ is the distance from the graph of g to the axis of revolution.

Examples - Find the surface of revolution formed by revolving

① $f(x) = x^2$ on $[1, 2]$ about y-axis

Sol.



$y = x^2, \quad 1 \le x \le 2$

$r(x) = x$

$$S = 2\pi \int_1^2 x\sqrt{1 + 4x^2}\, dx$$

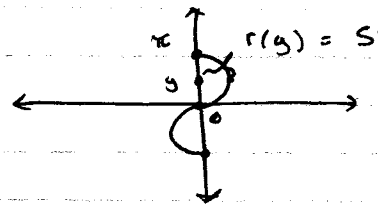
$t = 1 + 4x^2 \longrightarrow x = 1,\ t = 5$

$dt = 8x\, dx \qquad x = 2,\ t = 17$

$$= 2\pi \int_5^{17} \sqrt{t}\ \tfrac{1}{8}\, dt$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} t \Big|_5^{17} \Rightarrow \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$$

(2) $x = \sin(y)$ on $[0, \pi]$ about the y-axis.
$x = \sin(y)$, $0 \le x \le \pi$
$r(y)$



$$S = 2\pi \int_0^{\pi} \sin(y)\sqrt{1 + \cos^2(y)}\, dy$$

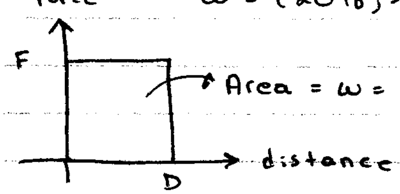
$t = \cos(y) \longrightarrow$ when $y = 0$, $t = 1$
$dt = -\sin(y)\, dx$ when $y = \pi$, $t = -1$

$$= 2\pi \int_1^{-1} \sqrt{1+t^2}\,(-1)\,dt = 2\pi \int_1^{-1} \sqrt{1+t^2}\,dt$$

NOT FINISHED.

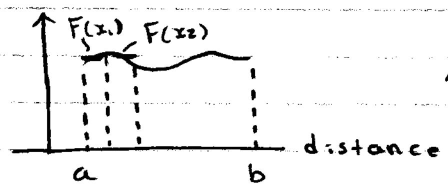
Work done by a constant force:
If an object is moved a distance D in the direction of a constant Force F, then the work done by the force is: $W = FD$

Example : Determine the work done by lifting a 20-pound weight 1 foot.
Sol. : $W = F \cdot D$
$W = (20\,\text{lb}) \cdot (1\,\text{ft}) \Rightarrow 20$ Foot . Pound



Area calculated by:

$$\lim_{\Delta D \to 0} \left(F(x_1)\Delta D + F(x_2)\Delta D + \dots + F(x_n)\Delta D \right)$$



Work done by a Variable Force
If a object is moved along a straight line by a continuous force $F(x)$ then the work done by the force as the object is moved from $x = a$ to $x = b$ is: $W = \int_a^b F(x)\,dx$

Hooke's Law

The Force F required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed or stretched from its original length that

$$F = kd$$

where k is the constant of proportionality that depends on the specific nature of the spring.

Example: A force of 600 pound compresses a spring 2 inches from its natural length of 10 inches. Find the work done in compressing the spring an additional 2 inches.

Sol: $F(x)$ = Force required to compress the spring by x units

So by Hooke's Law: $F(x) = kx$

$F(2) = 600$

$\therefore\ 600 = k(2)$

$k = 300$

So, $F(x) = 300x$

thus,

$$W = \int_2^4 F(x)\,dx = \int_2^4 300x\,dx$$

$$= 150x^2 \Big|_2^4 \Rightarrow 1800 \text{ in}\cdot\text{pounds}$$

CALC 2

1

JAN. 31/17

LAB 4 - Volume

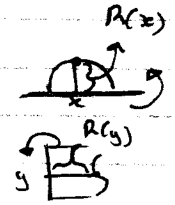
Disc Method

$V = \pi \int_a^b R(x)^2 dx$

Horizontal Axis

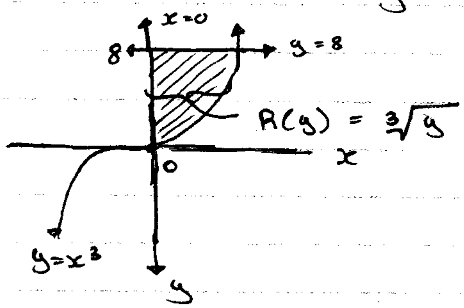
$V = \pi \int_c^d R(y)^2 dy$

Vertical Axis



① Find the volume of the solid obtained by rotating the region bounded by:

a) $y = x^3$, $y = 8$, and $x = \emptyset$ about the y-axis

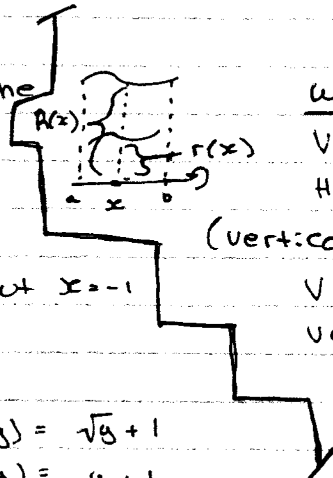


$0 \leq y \leq 8$

$$V = \pi \int_0^8 y^{2/3}\, dy = \pi\, 3/5\, y \Big|_0^8$$
$$= \frac{96\pi}{5}$$

② Find the volume of the solid obtained by rotating the region bounded by:

b) $y = x$, $y = x^2$, about $x = -1$



Washer Method

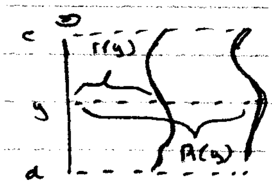
$V = \pi \int_a^b (R(x)^2 - r(x)^2)\, dx$

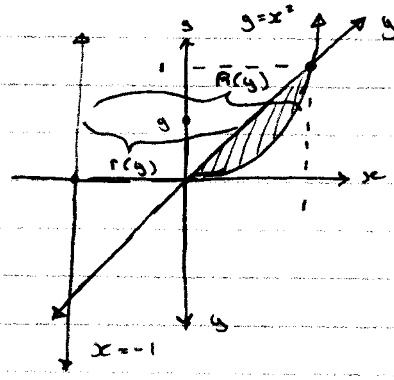
Horizontal Axis

(vertical axis is flipped)

$V = \pi \int_c^d (R(y)^2 - r(y)^2)\, dy$

Vertical axis



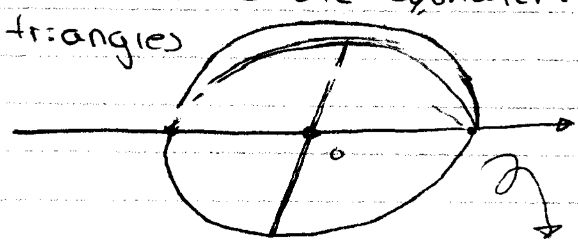


$R(y) = \sqrt{y} + 1$

$r(y) = y + 1$

$$V = \pi \int_0^1 (R(y)^2 - r(y)^2)\, dy = \pi \int_0^1 ((y^{1/2}+1)^2 - (y+1)^2)\, dy$$
$$= \pi \int_0^1 (y + 2y^{1/2} + 1 - y^2 - 2y - 1)\, dy$$
$$= \pi (4/3\, y^{3/2} - y^3/3 - y^2/2) \Big|_0^1$$
$$= \pi/2$$

③ Find the volume of the solid that has a circular base of radius 1 and whose parallel crossed-sections perpendicular to the base are equilateral triangles



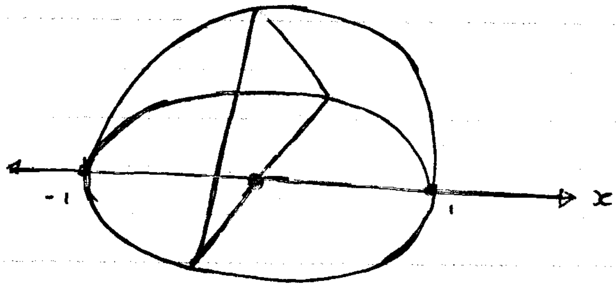
Cross-section Method

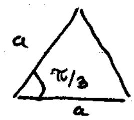
$V = \int_a^b A(x)\, dx$

Crossed-sections perpendicular to the x-axis

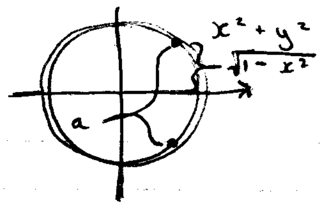
$V = \int_c^d A(y)\, dy$

Crossed-sections perpendicular to the y-axis





$$A = \tfrac{1}{2} a^2 \sin \pi/3 = \tfrac{\sqrt{3}}{4} a^2$$



$$a = 2\sqrt{1-x^2}$$

$$A(x) = \frac{\sqrt{3}}{4} \cdot \left(2\sqrt{1-x^2}\right)^2 = \frac{\sqrt{3}}{4} a^2$$

$$A(x) = \frac{\sqrt{3}}{4} \cdot \left(2 - \sqrt{1-x}\right)^2 = \sqrt{3}\,(1-x^2)$$

$$V = \int_{-1}^{1} (1-x) \ldots \Rightarrow \boxed{\frac{4\sqrt{3}}{3}}$$

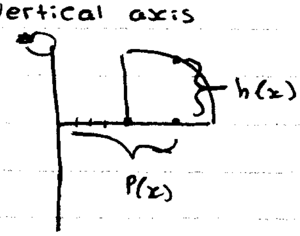
CAN'T READ, HE WON'T MOVE.

(1) $y = e^{-x^2}$, $y = 0$ about y-axis
$x = 0$, $x = 1$

Shell Method

$$V = 2\pi \int_a^b p(x)\,h(x)\,dx$$

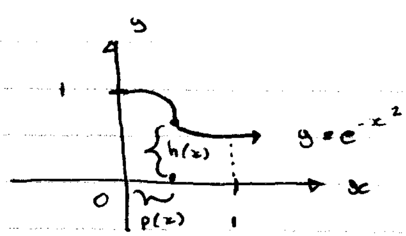
vertical axis



$$V = 2\pi \int_c^d p(y)\,h(y)\,dy$$

horizontal axis

Solution



$p(x) = x$
$h(x) = e^{-x^2}$

$$V = 2\pi \int_0^1 x e^{-x^2}\,dx$$

$$t = e^{-x}$$

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Lecture 11 • Work (Section 7.5)
• Moments, Center of Mass, Centroids (Sec. 7.6)

Work done by a Variable Force

- The work done by a continuous varying force, $F(x)$ as the object is moved from $x=a$ to $x=b$ in a straight line is

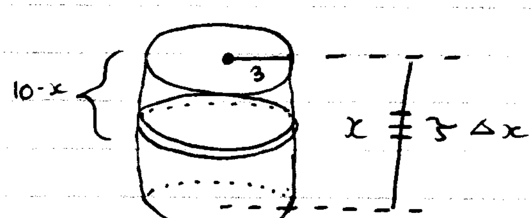
$$W = \int_a^b F(x)dx \Rightarrow \lim_{\Delta x \to 0} \left(F(x_1)\Delta x + F(x_2)\Delta x + \dots F(x_n)\Delta x \right)$$

(If the force is constant, that is $F(x) = F$ for all x, then $W = \int_a^b F dx = F \int_a^b dx = F(b-a)$
$= F \cdot D$)

Examples

① Find the work done by pumping out water from the top of a cylindrical tank 3 ft in radius and 10 ft tall, if the tank is initially full. (The density of water is 62.4 lb/ft³)

Solution



$F = (\text{Density}) \times \begin{pmatrix}\text{Volume of the}\\ \text{small cylinder}\end{pmatrix}$

$F = 62.4 \times \pi(3)^2 \Delta x$

$W_{\text{small cylinder}} = F \cdot \text{distance}$
$= 561.6\, \pi\, (10-x)\Delta x$

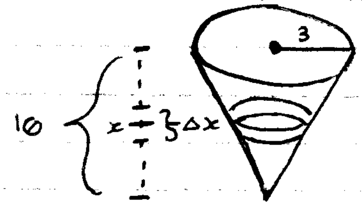
$$W = \lim_{\Delta x \to 0} \left(561.6\,\pi(10-x_1)\Delta x + 561.6\,\pi(10-x_2)\Delta x \dots + 561.6\,\pi(10-x_n)\Delta x \right)$$

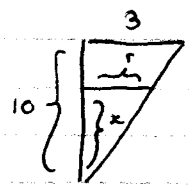
$$= \int_0^{10} 561.6\,\pi(10-x)\,dx = 561.6\,\pi \int_0^{10} (10-x)\,dx$$

$$= 561.6\,\pi \left(10x - \frac{x^2}{2}\right)\Big|_0^{10}$$

$$= 8.82 \cdot 10^4 \text{ ft-pound}$$

2. Find the work done by pumping out molasses from a conical tank filled to 2 ft from the top of the tank. The tank has a maximum radius of 3 ft and a height of 10 ft. Molasses weighs 100 lb/ft³





$$\frac{r}{3} = \frac{x}{10}$$

$$r = \frac{3x}{10}$$

$$\text{Force} = 100 \cdot \text{Volume} = 100 \cdot \pi r^2 \cdot \Delta x$$

$$= 100\,\pi \left(\frac{3x}{10}\right)^2 \cdot \Delta x$$

$$\text{Work} = 100\,\pi \frac{(3x)^2}{10} \Delta x\,(10-x)$$

$$W = \int_0^8 100\,\pi \left(\frac{3x}{10}\right)^2 (10-x)\,dx$$

$$= 1.930 \times 10^4 \text{ ft} \cdot \text{lb}$$

Moment and Center of mass : <u>One dimensional sys:</u>

Let the point masses $m_1, m_2, m_3, \ldots, m_n$ be located at $x_1, x_2, x_3, \ldots, x_n$

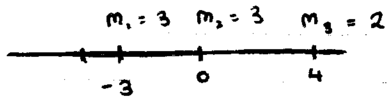
1. The moment about the origin is
$M_o = m_1 x_1 + m_2 x_2 + \ldots + m_n x_n$
2. The center of mass is: $\bar{x} = \dfrac{M_o}{m}$ (where $m = m_1 + m_2 + \ldots + m_n$) is the total mass

<u>Two dimensional system :</u>

1. The moment about the y-axis
$M_y = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n$
2. The moment about the x-axis
$M_x = m_1 y_1 + m_2 y_2 + \ldots m_n y_n$
3. The center of maes is $(\bar{x}, \bar{y})$
$\bar{x} = \dfrac{M_y}{m} \qquad \bar{y} = \dfrac{M_x}{m}$

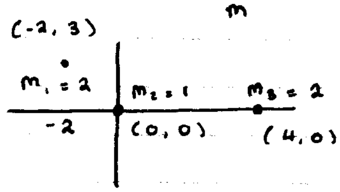
Examples:

1



$$M_0 = -3 \cdot 3 + 3 \cdot 0 + 2 \cdot 4 = -1$$
$$m = 3 + 3 + 2 = 8$$
$$\bar{x} = \frac{M_0}{m} = -1/8$$

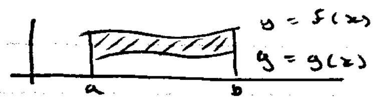
2



$$M_y = -2 \cdot 2 + 0 \cdot 1 + 2 \cdot 4 = 2$$
$$M_x = 3 \cdot 2 + 0 \cdot 1 + 0 \cdot 2 = 6$$
$$m = 2 + 1 + 2 = 5$$
$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{2}{5}, \frac{6}{5}\right)$$

Density: measure mass per unit volume

Planar Lamina: Thin, flat plate of material that has a constant uniform density of the form.



$f(x) = g(x)$ for all x

Mass moments and center of mass of Planar Laminus of uniform constant density ρ.

1 Mass
$$m = \rho \int_a^b (f(x) - g(x))\,dx \quad (\text{density} \times \text{area})$$

2 Moments abo

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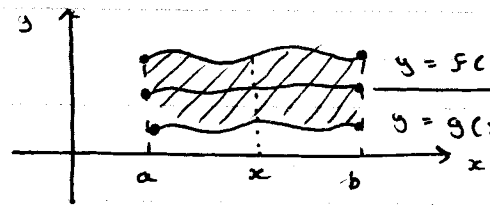
Lecture 12 - Moments, Center of mass and centroids (sec 7.6, cont)
- Fluid pressure and fluid Force (Sec. 7.7)

Mass $= \rho \int_a^b (f(x) - g(x))\,dx$ = density × Area

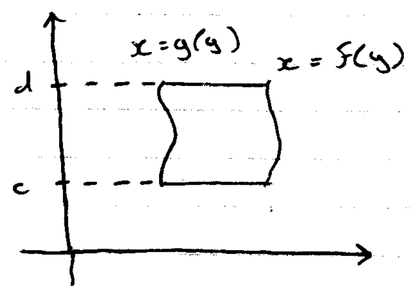


Moments about x and y axes.

$M_x = \rho \int_a^b \frac{f(x)^2 - g(x)^2}{2}\,dx = \rho \int_a^b \left(\frac{f(x) g(x)}{2}\right)\left(f(x) - g(x)\right) dx$

$\boxed{M_x = \iint_R \rho\, y\,dx\,dy}$

$\boxed{\begin{aligned} M_y &= \rho \int_a^b x(f(x) - g(x))\,dx \\ &= \iint_R \rho\, x\,dx\,dy \end{aligned}}$



mass $= \rho \int_c^d (f(y) - g(y))\,dy$

Center of mass

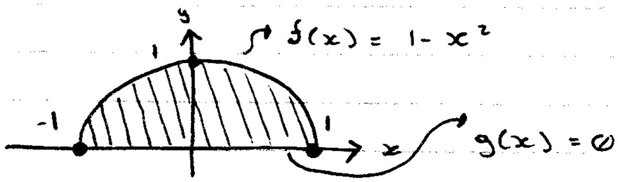
$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$

Moments about x and y axes

$M_x = \rho \int_c^d y\,(f(y) - g(y))\,dy = \iint_R \rho\, y\,dx\,dy$

$M_y = \rho \int_c^d \left(\frac{f(y)^2 - g(y)^2}{2}\right) dy = \iint_R \rho\, x\,dx\,dy$

Examples : Find the center of mass of a lamina of uniform density $\rho = 2$ bounded by the graphs of (1) $f(x) = 1 - x^2$ and the x-axis



Sol: $1 - x^2 = 0$

$x = \pm 1$

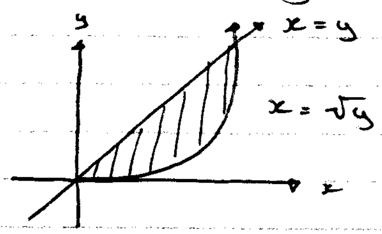
$$m = 2\int_{-1}^{1} (1 - x^2)\,dx = 2\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = 8/3$$

$$M_x = 2\int \frac{(1-x)^2}{2}\,dx = \int_{-1}^{1} (1 - 2x^2 + x^4)\,dx = \left(x - \frac{2x^3}{3} + \frac{x^5}{5}\right)\Big|_{-1}^{1} = 16/15$$

$$M_y = 2\int_{-1}^{1} x(1 - x^2)\,dx = 2\int_{-1}^{1}\left(x - \frac{x^3}{3}\right)dx = 0$$

(because function is odd.)

Question: Find the center of mass of a lamina of uniform density $\rho = 2$ bounded by the graphs of ① $x = y$ and $x = \sqrt{y}$



Sol: $y = \sqrt{y}$

$y^2 = y$

$y(y-1) = 0$

$y = 0$ or $y = 1$

$$m = 2\int_0^1 (\sqrt{y} - y)\,dy = 2\int_0^1 (y^{1/2} - y)\,dy$$
$$= 2\left(2/3\, y^{2/3} - y^2/2\right)\Big|_0^1 = \boxed{1/3}$$

$$M_x = 2\int_0^1 y(\sqrt{y} - y)\,dy = 2\int_0^1 y^{3/2} - y^2\,dy$$
$$= 2\left(2/5\, y^{5/2} - y^3/3\right)\Big|_0^1 = \boxed{2/15}$$

$$M_y = 2\int_0^1 \frac{y - y^2}{2}\,dy = \left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 = \boxed{1/6}$$

$$\text{Center of mass} = (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$
$$= (1/2, 2/5)$$

Pressure - Force per unit area over the surface of a body.

Fluid Force $= F = P \cdot A = (\text{Pressure}) \cdot (\text{Area})$

Fluid Pressure - The pressure on an object at depth h in a liquid is Pressure $= P = wh$ where w is the weight density per unit.

Water $\quad w = 62.4 \text{ lb/ft}^3$

Example: Find the fluid force on a rectangular metal sheet measuring 3 feet by 4 feet that is submerged in 6 feet of water.

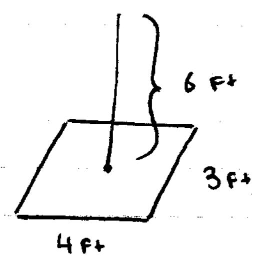
Solution:

$F = P \cdot A$

$= whA$

$= (62.4)(6)(4 \cdot 3)$

$= 4492.8 \text{ lb}$

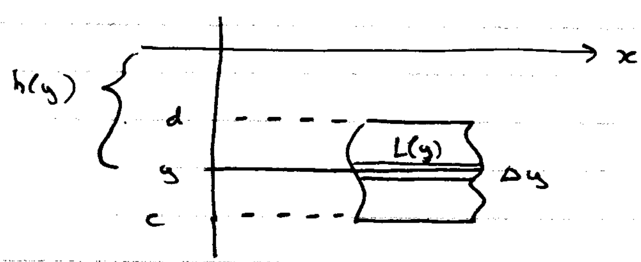


Force exerted by a Fluid:

The force exerted by a fluid of a constant weight density w (per unit of volume) against a submerged vertical plane region from $x = c$ to $y = d$ is

$$F = w\int_c^d h(y)\,L(y)\,dy$$
$$= w \lim_{\Delta y \to 0} \left(h(y_1)L(y_1) + \dots + h(y_n)L(y_n) \right)$$

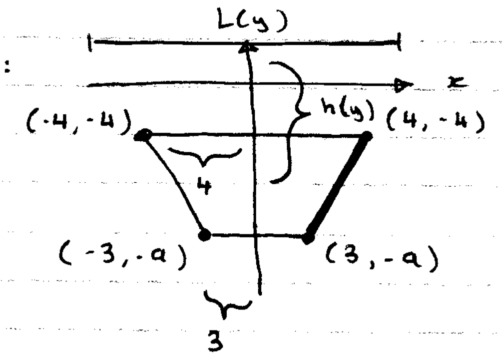
where $h(y)$ is the depth of the fluid at y and $L(y)$ is the horizontal length of the region at y.



$F(y) = P(y)A = wh(y)L(y)\Delta y$

A vertical gate in a dam has the shape of an iscoceles trapezoid 8 feet across the top and 6 feet across the bottom with a height of 5 feet. What is the Fluid Force on the gate when the top of the gate is 4 feet below the surface of the water?

Sol:



$h(y) = -y$

$(y+4) = (5/1)(x-4)$

$x = 4 + 1/5(y+4)$

$L(y) = 2(4 + (1/5)(y+4))$

$h(y) = -y$

$a = -9$

$b = -4$

$$\int_{-9}^{-4} -y(2(4+(1/5)(y+4)))$$

$= 13936$ lb