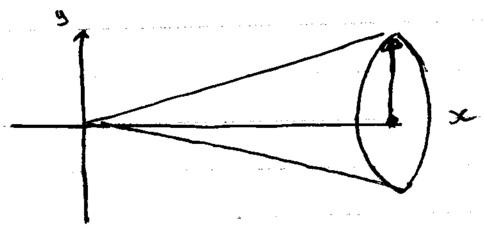
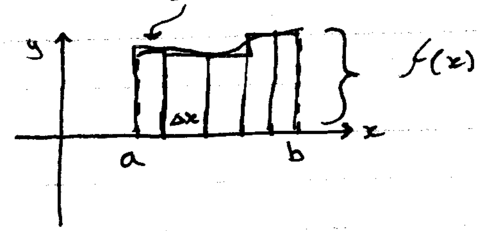
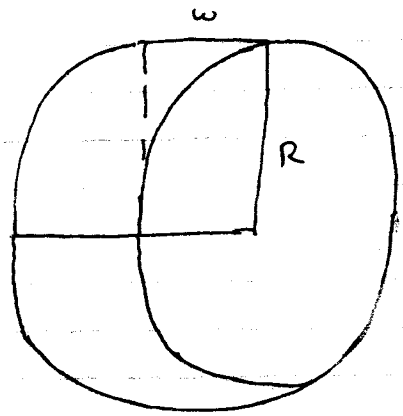
# Lecture 7 - Volume (Section 7.2)

- The disk method
- The washer
- Crossed - section




$V = \pi R^2 w$

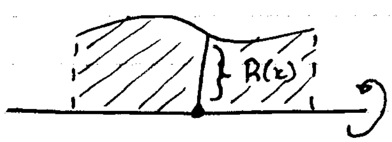
$\lim_{\Delta x \to 0} \left( \pi f(x_1)^2 \Delta x + \pi f(x_2)^2 \Delta x + \ldots \ldots \pi f(x_n)^2 \Delta x \right)$

$\Downarrow$

$\int_a^b \pi f(x)^2 dx$

Thm: The disk method

Volume $= V = \pi \int_a^b R(x)^2 dx$



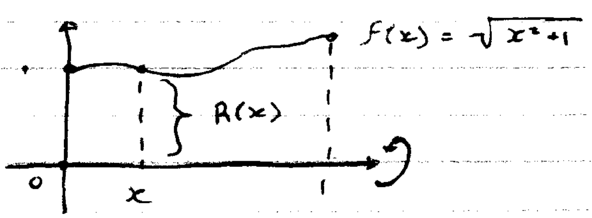

Horizontal axis of revolution.

Volume $= V = \pi \int_c^d R(y)^2 dy$


Examples :

Find the volume of the solid formed by revolving the region bounded to the graph of :

1. $f(x) = \sqrt{x^2+1}$, $0 \le x \le 1$ about the x-axis



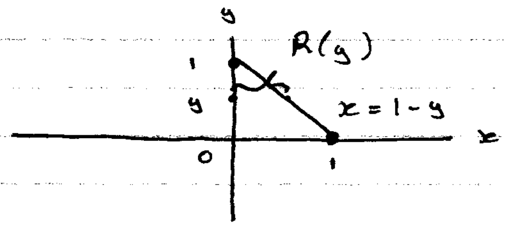

$R(x) = \sqrt{x^2+1}$

Volume $= \pi \int_0^1 R(x)^2 dx = \pi \int_0^1 (x^2+1) dx$

$= \pi \left( \frac{x^3}{3} + x \right) \Big|_0^1 = \frac{4\pi}{3}$

2 $f(y) = 1 - y \quad 0 \leq y \leq 1$ about the $y$-axis
$(x = 1 - y)$

Solution




$R(y) = 1 - y$

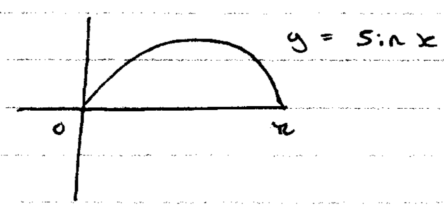
$$\text{Volume} = \pi \int_0^1 R(y)^2 \, dy = \pi \int_0^1 (1-y)^2 \, dy$$

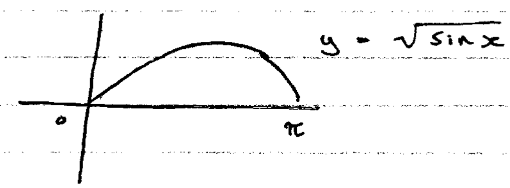
$$= \pi \int_0^1 (1 - 2y + y^2) \, dy$$

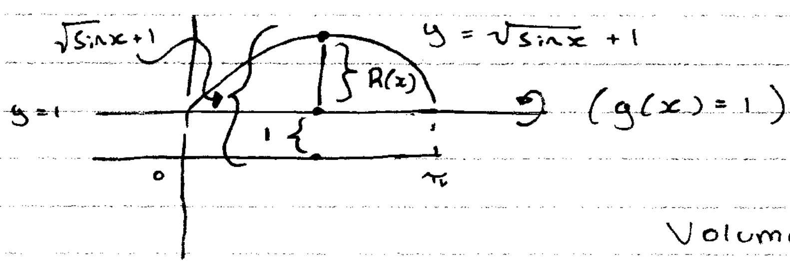
$$= \pi \left( y - y^2 + \frac{y^3}{3} \right) \Big|_0^1 = \frac{\pi}{3}$$

3 $f(x) = \sqrt{\sin x} + 1, \quad g(x) = 1$ about the $y$-axis
$0 \leq x \leq \pi$

Solution




$R(x) = \sqrt{\sin x} + 1 - 1$
$= \sqrt{\sin x}$

$$\text{Volume} = \pi \int_0^\pi R(x)^2 \, dx$$

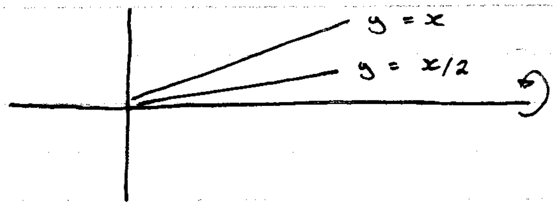
$$= \pi \int_0^\pi \sin x \, dx$$

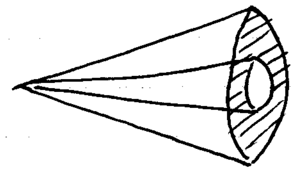
$$= \pi \int_0^\pi (-\cos) \Big|_0^\pi$$

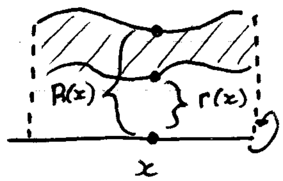
$$= (-\pi \cos \pi + \pi \cos 0)$$

$$= \pi + \pi = 2\pi$$

Thm: The washer method:

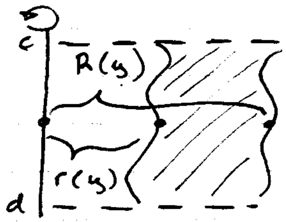





$$\text{Volume} = V = \pi \int_a^b R(x)^2\,dx - \pi \int_a^b r(x)^2\,dx$$

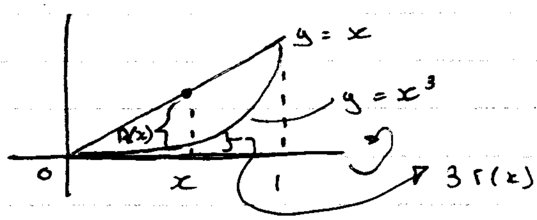
$$V = \pi \int_a^b \left(R(x)^2 - r(x)^2\right) dx$$




$$\text{Volume} = V = \pi \int_c^d \left(R(y)^2 - r(y)^2\right) dx$$

Examples:

1. $y = x$ and $y = x^3$ about the $x$-axis
   $0 \le x \le 1$

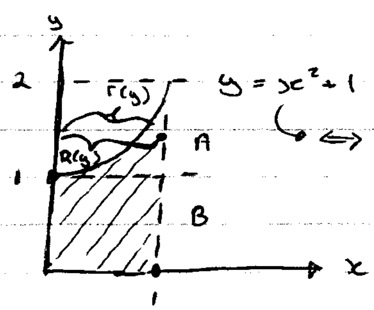



$r(x) = x^3$
$R(x) = x$

$$V = \pi \int_0^1 \left(R(x)^2 - r(x)^2\right) dx$$

$$= \pi \int_0^1 (x^2 - x^6)\,dx = \pi\left(\frac{x^3}{3} - \frac{x^7}{7}\right)$$

2. $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$
   about the $y$-axis




$r(y) = \sqrt{y-1}$
$R(y) = 1$

$$V_A = \pi \int_1^2 \left(R(y)^2 - r(y)^2\right) dy$$

$$= \pi \int_1^2 \left(1^2 - (\sqrt{y-1})^2\right) dy$$

(A)

$$= 2\pi - 3/2\,\pi = \pi/2$$

B $\quad r(y) = 0 \qquad V_B = \pi \int_0^1 (1^2 - 0^2)\, dy = \pi$

$R(y) = 1$

$$\begin{aligned} V_{TOTAL} &= V_A + V_B \\ &= \pi/2 + \pi \\ &= \tfrac{3}{2}\pi \end{aligned}$$

1. Evaluate
   a) $\sinh(\ln 4)$
   b) $\tanh(\ln 2)$
   c) $\sinh^{-1}(2)$
   d) $\operatorname{sech}^{-1}(1/2)$

Remember that:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$$

$$\operatorname{sech}^{-1}(x) = \ln \frac{1 + \sqrt{1-x^2}}{x}$$

a) $\sinh(\ln 4)$

$$= \frac{e^{\ln 4} - e^{-\ln 4}}{2}$$

$(-\ln 4 = \ln(4^{-1}) = \ln(1/4))$

$$= \frac{4 - (1/4)}{2}$$

$$\boxed{= \frac{15}{8}}$$

b) $\tanh(\ln 2)$

$$= \frac{e^{\ln 2} - e^{-\ln 2}}{e^{\ln 2} + e^{-\ln 2}}$$

$$= \frac{2 - (1/2)}{2 + (1/2)}$$

$$\boxed{= \frac{3}{5}}$$

c) $\sinh^{-1}(2)$

$$\ln(2 + \sqrt{2^2+1})$$

$$\boxed{= \ln(2 + \sqrt{5})}$$

d) $= \operatorname{sech}^{-1}(1/2)$

$$= \ln \frac{1 + \sqrt{1-(1/2)^2}}{1/2}$$

$$\boxed{= \ln(2 + \sqrt{3})}$$

2. Show $\sinh^2 x = \cosh^2 x - 1$

$$\sinh^2 x = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} - e^{-2x} - 2}{4}$$

$$\cosh^2 x - 1 = \left(\frac{e^x + e^{-x}}{2}\right)^2 - 1 = \frac{e^{2x} + e^{-2x} + 2}{4} - 1$$

$$= \frac{e^{2x} + e^{-2x} - 2}{4}$$

$=$

3. Find $\lim_{x \to \infty} \tanh x$ :

$$\Rightarrow \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{1/e^x}{1/e^x}$$

(Note: $\lim_{x\to\infty} e^{-x} \Rightarrow 1/e^x$)

$$\Rightarrow \lim_{x\to\infty} \frac{1 - (e^{-x})^2}{1 + (e^{-x})^2}$$

$$\lim_{x\to\infty} \frac{1 - \emptyset}{1 + \emptyset}$$

$$\boxed{\Rightarrow 1}$$

4. Find the equation of the tangent line to the graph of $f(x) = \ln(\tanh(x/2))$ at $x = \ln 4$

Solution:

Point $\Rightarrow (\ln 4, f(\ln 4)) \Rightarrow \left(\ln 4, \ln(3/5)\right)$

Slope $\Rightarrow f'(\ln 4)$

$$f(\ln 4) = \ln\left(\tanh\left(\frac{\ln 4}{2}\right)\right) = \ln\left(\tanh\left(\ln 4^{1/2}\right)\right)$$
$$= \ln(\tanh(\ln 2))$$
$$= \ln(3/5)$$

$$f'(x) = \frac{1}{\tanh(x/2)} \cdot (\operatorname{sec}^2 h(x/2)) \cdot 1/2$$
$$f'(\ln 4) = \frac{1}{\tanh\left(\frac{\ln 4}{2}\right)} \cdot \left(\operatorname{sec}^2 h\left(\frac{\ln 4}{2}\right)\right) \cdot \frac{1}{2}$$
$$\Rightarrow \frac{5}{6} \operatorname{sec}^2 h(\ln 2)$$
$$\Rightarrow 5/6 \cdot (4/5)^2 = \boxed{8/15}$$

5. $\int \operatorname{sech}^3 x \tanh x \, dx \Rightarrow \int \operatorname{sec}^3 hx \cdot u \cdot \frac{du}{\operatorname{sec}^2 hx}$

$u = \tanh x$ $\Rightarrow \int \operatorname{sech} x \cdot u \cdot du$

$du = \operatorname{sec}^2 h x \, dx$

$dx = \frac{du}{\operatorname{sec}^2 h x}$ $\Rightarrow \operatorname{sech} x \int u \cdot du$ X WRONG!

NOTE: $(\tanh x)' = \operatorname{sech}^2 x$
$(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$

use $t = \operatorname{sech} x$
$dt = -\operatorname{sech} x \tanh x$

$$\Rightarrow \int \operatorname{sech}^3 x \tanh x \, dx \Rightarrow \int \frac{\operatorname{sech}^2 x}{t^2} \cdot \frac{\operatorname{sech} x \tanh x}{-dt}$$
$$\Rightarrow \int t^2 \cdot (-1)\, dt$$
$$\Rightarrow -\int t^2 dt \Rightarrow$$ and then he erased everything.
(but it's almost done)

b) $\int_0^1 \frac{1}{4-x^2}\,dx$ c) $\int \frac{1}{x\sqrt{1-x^4}}\,dx$

Using

① $\int \frac{u'\,dx}{\sqrt{u^2 \pm a^2}} = \ln\left(u + \sqrt{u^2+a^2}\right) + C$

② $\int \frac{u'\,dx}{a^2-u^2} = \frac{1}{2a} \ln\left|\frac{a+u}{a-u}\right| + C$

③ $\int \frac{u'\,dx}{u\sqrt{a^2 \pm u^2}} = \frac{1}{a} \ln\ a \pm$

b) $\int_0^1 \frac{1}{4-x^2}\,dx \Rightarrow \int_0^1 \frac{1}{2^2-x^2}\,dx \Rightarrow \left(\frac{1}{4}\right) \ln\left|\frac{2+x}{2-x}\right|\Big|_0^1$

$a = 2$

$u = x$

c)

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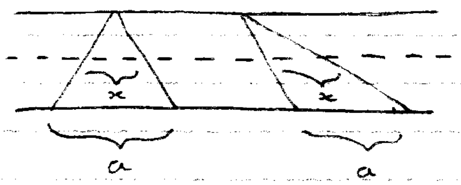
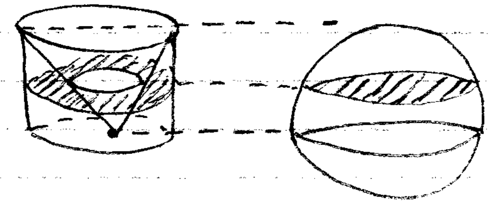
Lecture 8 - Volume

↳ the cross section method (section 7.2)

↳ the shell method (section 7.3)

Cavalieri's Principle



Thm: (the cross-sectional method)

Let R be a solid

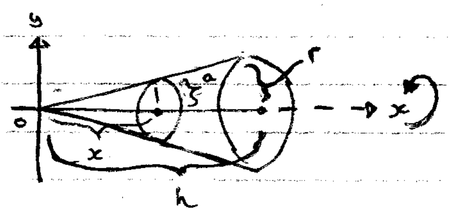
① For cross-sections of area $A(x)$ taken perpendicular to the x-axis $(a \le x \le b)$ The volume of R is given by: $V = \int_a^b A(x)\,dx$

② For cross-section of area $A(y)$ of

$$V = \int_c^d A(y)\,dy$$

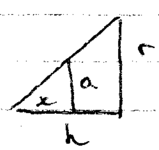
Examples

① Find the formula for the volume of right cone




$0 \le x \le h$

$A(x) = \pi\left(\frac{rx}{h}\right)^2 = \frac{\pi r^2}{h^2} \cdot x^2$




$\frac{a}{r} = \frac{x}{h}$

$\Rightarrow a = \frac{rx}{h}$

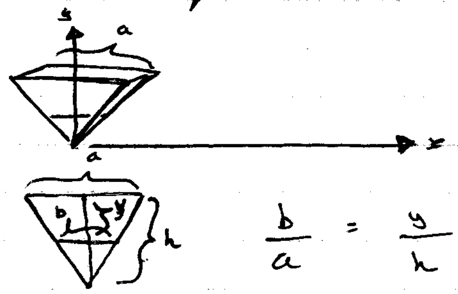
$$V = \int_0^h \frac{\pi r^2}{h^2} x^2\,dx$$

$$= \frac{\pi r^2}{h^2} \int_0^h x^2\,dx$$

$$= \frac{\pi r^2}{h^2} \cdot \frac{x^3}{3} \Big|_0^h$$

$$= \frac{\pi r^2}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3}\pi r^2 h$$

(2) Find the Formula for the volume of Pyramid of Square base $0 \le y \le h$




$$\frac{b}{a} = \frac{y}{h}$$

$$\Rightarrow b = \frac{ya}{h}$$

$$A(y) = \frac{y^2 a^2}{h^2}$$

$$V = \int_0^h A(y)\,dy = \int_0^h \frac{y^2 a^2}{h^2}\,dy$$

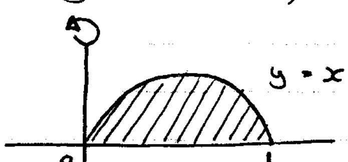
$$\Rightarrow \frac{a^2}{h^2}\int_0^h y^2\,dy$$

$$\Rightarrow \frac{a^2}{h^2}\cdot\frac{y^3}{3}\Big|_0^h$$

$$\Rightarrow \frac{1}{3}a^2 h$$

Example: Find the Volume of the Volume of the Solid formed by revolving the region bounded by:

(1) $y = x - x^4$, $0 \le x \le 1$ about the $y$-axis




Solution: $2\pi\int_0^1 p(x)h(x)\,dx$

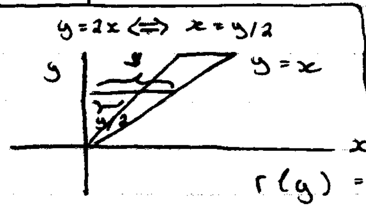
$$= 2\pi\int_0^1 x(x - x^4)\,dx$$

$$= 2\pi\int_0^1 (x^2 - x^5)\,dx$$

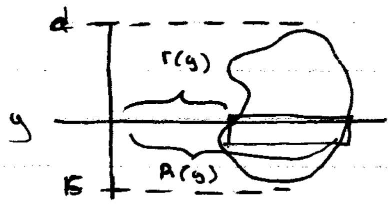
$$= 2\pi\left(\frac{x^3}{3} - \frac{x^6}{6}\right)\Big|_0^1$$

$$\Rightarrow \pi/3$$

$y = 2x \Leftrightarrow x = y/2$




$r(y) = y/2$ $\quad R(y) = y$



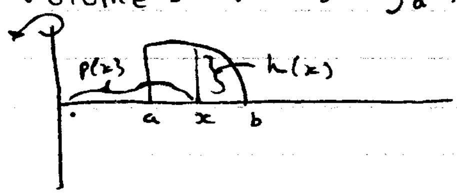

$$V = \pi\int_c^d \left(R(y)^2 - r(y)^2\right)dy$$

(washer method)

(shell method)

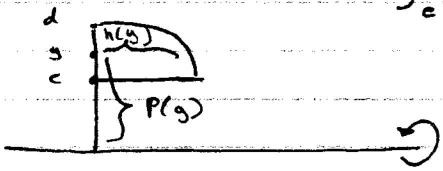
Thm Vertical axis of revolution

Volume = $V = 2\pi\int_a^b p(x)\,h(x)\,dx$



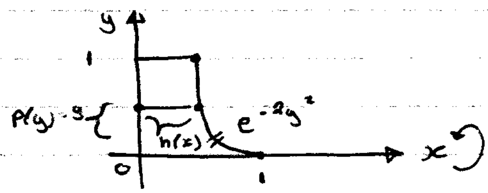
Horizontal axis of revolution:

Volume = $V = 2\pi \int_c^d P(y)\, h(y)\, dy$




Examples:

(2) $x = e^{-2y^2}$, $0 \le y \le 1$ about the x-axis




$0 \le y \le 1$

$P(y) = y$

$h(y) = e^{-2y^2}$

$V = 2\pi \int_0^1 P(y)\, h(y)\, dy$

$2\pi \int_0^1 P(y)\, h(y)\, dy = 2\pi \int_0^1 y e^{-2y^2} dy$

$t = -2y^2$

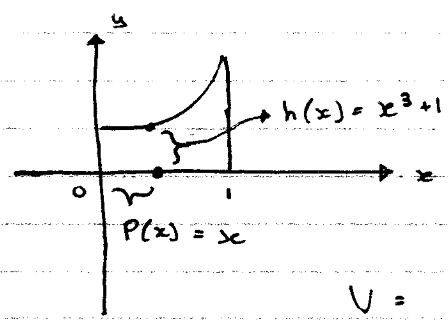
$dt = -4y\, dy$

$y\, dy = \frac{-1}{4} dt$

$\Rightarrow \frac{\pi}{2} \int_{-2}^{0} e^t dt = \frac{\pi}{2} e^t \Big|_{-2}^{0}$

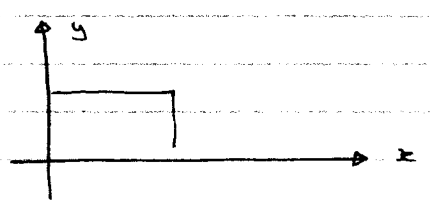
$\Rightarrow 2\pi \int_0^{-2} e^t (-\tfrac{1}{4})\, dt$

(3) $y = x^3 + 1$, $y = 0$, $x = 0$, $x = 1$




$V = 2\pi \int_0^1 x(x^3 + 1)\, dx$

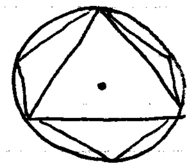
Using washer method:




(finish from photo.)

JAN. 27/17

# Lecture 9 - Arc length and Surfaces of Revolution (Section 7.4)



Length = $2\pi$

**Def'n** (smooth curves)
Let $f$ be a differentiable with continuous derivative on $[a, b]$. Then the graph of $f$ is called a smooth curve.

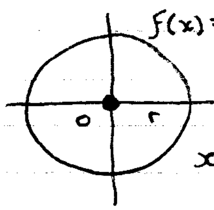
· **Thm** (Arc Length)
Let $y = f(x)$ represent a smooth curve on $[a, b]$. The arc length of $f$ between $a$ and $b$ is

$$S = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

Similarly, for a smooth curve $x = g(y)$ is

$$S = \int_c^D \sqrt{1+(g'(y))^2}\,dy$$

**Examples**
Find the formula for the arc length of a circle.
**Solution**




$f(x) = \sqrt{r^2 - x^2}$

Arc-length = 4

$x^2 + y^2 = r^2$

$\Rightarrow 4r \cdot \frac{\pi}{2} = 2r\pi$

$f(x) = \sqrt{r^2 - x^2}$ (r is a constant)

$f'(x) = (1/2)(r^2 - x^2)^{-1/2}(-2x)$

$= \frac{-\cancel{2}x}{\cancel{2}(r^2 - x^2)^{1/2}} \Rightarrow -\frac{x}{\sqrt{r^2 - x^2}}$

$f'(x) = \frac{-x}{\sqrt{r^2 - x^2}} \Rightarrow (f'(x))^2 = \frac{x^2}{r^2 - x^2}$

$1 + (f'(x))^2 = \frac{x^2}{r^2 - x^2} + 1$

$\Rightarrow \frac{r^2}{r^2 - x^2}$

Find the arc length of the graph of

a) $f(x) = \frac{x^4}{8} + \frac{1}{4x^2}$ on $[1,2]$

Sol

$$S = \int_1^2 \sqrt{1 + (f'(x))^2}\, dx$$

$$f'(x) = \frac{x^3}{2} - \frac{1}{2x^3} = \frac{1}{2}(x^3 - 1/x^3)$$

$$= \int_1^2 \sqrt{1/4 (x^3 + 1/x^3)^2}\, dx$$

$$(f'(x))^2 = 1/4 \left(x^6 + \frac{1}{x^6} - 2\right)$$

$$= \int_1^2 1/2 \, (x^3 + x^{-3})\, dx$$

$$(f'(x))^2 + 1 = 1/4 (x^6 + 1/x^6 - 2 + 4)$$
$$= 1/4 (x^3 + 1/x^3)^2$$

$$= 1/2 \left(x^4/4 - 1/2x^2\right)\Big|_1^2$$

$$33/32$$

b) $y^3 = x^2$ for $0 \le x \le 8$

① $y = x^{2/3}$, $0 \le x \le 8$

$$S = \int_0^8 \sqrt{1 + (2/3\, x^{-1/3})^2}\, dx = \int_0^8 \sqrt{1 + 4/9\, x^{-2/3}}\, dx$$

② $x = y^{3/2}$

$x = 0 \Rightarrow y = 0^{2/3} \Rightarrow 0$

$x = 8 \Rightarrow y = 8^{2/3} \Rightarrow 4$

$0 \le y \le 4$

Arc length =

$$2\int_0^4 \sqrt{1 + (3/2\, y^{1/2})^2}\, dy = 2\int_0^4 \sqrt{1 + 9/4\, y}\, dy$$

$t = 1 + 9/4\, y$

$dt = 9/4\, dy$

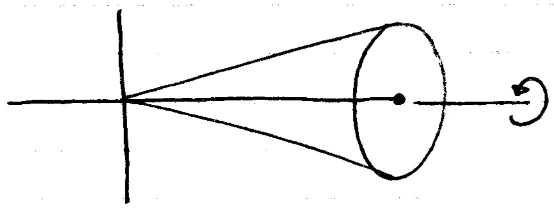
$y = 0 \Rightarrow t = 1$

$y = 4 \Rightarrow t = 10$

$$= 2\int_1^{10} \sqrt{t}\ 4/9\, dt$$

$$= 8/9 \int_1^{10} t^{1/2}\, dt = 8/9 \ 2/3\, t^{3/2}\Big|_1^{10}$$

$$= \frac{16}{27}\left(10^{3/2} - 1\right)$$

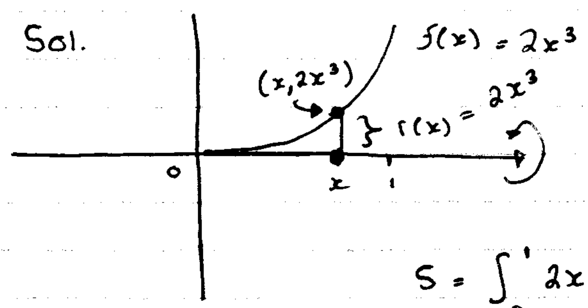


Thm (area of surface of revolution)
Let $y = f(x)$ represent a smooth curve on $[a, b]$.
Then the area $S$ of the surface of revolution formed by revolving the of $f$ about a horizontal or vertical axis is

$$S = \int_a^b r(x)\sqrt{1 + (f'(x))^2}\, dx$$

where $r(x)$ is the distance from the graph of $f$ to the axis of revolution.

Sol.




$$S = \int_0^1 2x^3\sqrt{1 + (6x^2)^2}\, dx$$

$$= 2\int_0^1 x^3\sqrt{1 + 36x^4}\, dx$$

$$t = 1 + 36x^4$$
$$dt = 144x^3\, dx$$

$$= 2\int_1^{37} \sqrt{t} = \frac{1}{144}\, dt = \frac{1}{72}\ \frac{2}{3} t^{3/2}\Big|_1^{37} = \frac{1}{108}\left(37^{3/2} - 1\right)$$