Lecture 4 - Hyperbolic Function (section 5.9)

Def'n

$\sinh x = \dfrac{e^x - e^{-x}}{2}$ $\qquad$ $\operatorname{csch} t = \dfrac{1}{\sinh x}$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$ $\qquad$ $\operatorname{sech} t = \dfrac{1}{\cosh x}$

$\tanh x = \dfrac{\sinh x}{\cosh x}$ $\qquad$ $\operatorname{Cot} = \dfrac{\cosh x}{\sinh x}$

$\boxed{\text{Complex numbers}}$

$i^2 = -1$ $\qquad$ $\mathbb{C} = \left\{ a+bi \;\middle|\; \begin{array}{l} a, b \in \mathbb{R} \\ i^2 = -1 \end{array} \right\}$

$y = \sin x, \quad x \in \mathbb{R}$

$\sin z = ?$

$\cos z = ?$

$\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$

$\cos(x)$

$x \in \mathbb{R}$

$\cos(ix) = \cosh x$

Thm

$\cosh^2 x - \sinh^2 x = 1$

$\tanh^2 x = 1 - \operatorname{sech}^2 x$

$\coth^2 x = 1 + \operatorname{csch}^2 x$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x+y) = \cosh x \sinh y + \sinh x \cosh y$

Thm (Derivatives and integrals of hyperbolic functions)

$d/dx\,(\sinh x) = \cosh x$

$d/dx\,(\cosh x) = \sinh x$

$d/dx\,(\tanh x) = \operatorname{sech}^2 x$

$d/dx\,(\coth x) = -\operatorname{csch}^2 x$

$d/dx\,(\operatorname{sech} x) = \cdot \operatorname{sech} x \tanh x$

$d/dx\,(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

| u differentiability |

$\int \cosh x \, dx = \sinh x + c$ | $\int (\cosh u) u' \, dx = \sinh u + c$

$\int \sinh x \, dx = \cosh x + c$ | $\int (\sinh u) u' \, dx = \cosh u + c$

$\int \operatorname{sech}^2 x \, dx = \tanh x + c$

$\int \operatorname{csch}^2 x \, dx = -\coth x + c$

$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$

$\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + c$

$c \in \mathbb{R}$

Proof

$d/dx \, (\sinh x)$

$\Rightarrow d/dx \left(\dfrac{e^x - e^{-x}}{2} \right)$

$\Rightarrow \frac{1}{2} \, d/dx \left(e^x - e^{-x} \right)$

$\Rightarrow \frac{1}{2} \left(e^x - e^{-x}(-1) \right)$

$\Rightarrow \frac{1}{2} \left(e^x + e^{-x} \right)$

$= \cosh x$ ∎

Examples

① $d/dx \left(\sinh(x^3 + x + 1) \right)$

$= \cosh(x^3 + x + 1) \cdot (x^3 + x + 1)'$

$= (3x^2 + 1) \cosh(x^3 + x + 1)$

② $d/dx \left(\ln(\cos x) \right) = \dfrac{1}{\cosh x} \cdot \sinh x$

$= \tanh x$

③ $\int \cosh(4x) \sinh^2(4x) \, dx$

$t = \sinh(4x)$

$dt = \cosh(4x) \cdot 4x$

$= \int t^2 \, \frac{1}{4} \, dt$

$= \dfrac{t^3}{12} + c$

$= \dfrac{\sinh^3(4x)}{12} + c$ where, $c \in \mathbb{R}$

/ $\int \cos(4x) \sin^2(4x) \, dx$

$t = \sin(4x)$

④ $\int x^2 \operatorname{sech}^2(x^3) \, dx$

$t = x^3 \Rightarrow dt = 3x^2 dx$

$\int \operatorname{sech}^2 t \, \frac{1}{3} \, dt$

$= \frac{1}{3} \tanh t + c$

$= \frac{1}{3} \tan(x^3) + c, \; c \in \mathbb{R}$

5 $\int_0^{\ln 2} 2e^{-x}\cosh x\,dx$

$= \int_0^{\ln 2} \cancel{2}e^{-x}\left(\frac{e^{x}+e^{-x}}{\cancel{2}}\right)dx$

$= \int_0^{\ln 2} (1+e^{-2x})\,dx$

$\Rightarrow x\Big|_0^{\ln 2} + \int_0^{\ln 2} e^{-2x}\,dx$

$\Rightarrow \ln 2 - 0 + \frac{e^{-2x}}{-2}\Big|_0^{\ln 2}$

$\Rightarrow \ln 2 - \tfrac{1}{2}(e^{-2\ln 2} - e^{0})$

$\Rightarrow \ln 2 + 7/8$

$\int_0^{\ln 2} e^{-2x}dx \Rightarrow \int_0^{-2\ln 2} e^{t}(-\tfrac{1}{2})\,dt$

$t = -2x$

$dt = -2\,dx$

$x = 0 \Rightarrow t = 0$

$x = \ln 2 \Rightarrow t = -2\ln 2$

$= -\tfrac{1}{2}e^{t}\Big|_0^{-2\ln 2}$

$= -\tfrac{1}{2}(e^{-2\ln 2} - e^{0})$

Note: $e^{\ln a} = a$

So, $e^{-2\ln 2} = 1/4$

LAB 2 - Log rule and integrals involving inverse trigonometric Functions (odd and even Functions)

① $\int \frac{1}{x \ln(\sqrt[3]{x})} dx$ Using Log rule $\left[\int \frac{u'(x)}{u(x)} dx = \ln|u(x)| + C \quad C \in \mathbb{R} \right]$

$\ln(\sqrt[3]{x}) = \ln(x^{1/3}) = \frac{1}{3} \ln x$

$\Rightarrow 3 \int \frac{1}{x \ln x} dx \Rightarrow 3 \int \frac{(1/x)}{\ln x} dx = 3 \ln|\ln x| + C \quad C \in \mathbb{R}$

$u(x) = \ln x$

② $\int \frac{x^4 + 3x^2 + 2x}{x^2 + 1} dx$

$\Rightarrow \int \left(x^2 + 2 + \frac{2x - 2}{x^2 + 1} \right) dx$

$\Rightarrow \frac{x^3}{3} + 2x + \int \frac{2x}{x^2+1} dx - \int \frac{2}{x^2+1} dx$

$\Rightarrow \frac{x^3}{3} + 2x + \ln|x^2+1| - 2\arctan x + C \quad C \in \mathbb{R}$

Long division:

$$\begin{array}{r l}
 & x^2 + 2 \\
x^4 + 3x^2 + 2x & \big) \; x^2 + 1 \\
- & x^4 + x^2 \\
\hline
 & \varnothing + 2x^2 + 2x \\
- & + 2x \qquad + 2 \\
\hline
 & \varnothing + 2x - 2
\end{array}$$

③ $\int_0^1 f(x^4 + \cos x) dx = 1$

Let $g(x) = f(x^4 + \cos x)$

we have $g(-x) = f((-x)^4 + \cos(-x))$

$= f(x^4 + \cos x)$

$= 2(x)$

compute $\int_{-1}^{1} f(x^4 + \cos x) dx$

So g is even and so

$\int_{-1}^{1} f(x^4 + \cos x) dx$

$\Rightarrow \int_{-1}^{1} g(x) dx = 2 \int_0^1 g(x) dx$

$\Rightarrow 2 \int_0^1 f(x^4 + \cos x) dx$

$\Rightarrow 2^{\circ}$ (check this)

④ $\int_{-1}^{1} \frac{\tan(x^3 + x) + \cot(x^3 + x)}{x^8 + \sec x} dx$

Let $f(x) = \frac{\tan(x^3 + x) + \cot(x^3 + x)}{x^8 + \sec x}$

then $f(-x) = \frac{\tan(-x^3 - x) + \cot(-x^3 - x)}{-x^8 + \sec(-x)}$

$\Rightarrow \frac{-\tan(x^3 + x) - \cot(x^3 + x)}{x^8 + \sec(x)} = -f(x)$

Hence f is odd so the integral is $\varnothing$

5 $\int_{-1}^{1} \frac{dx}{x^2+8x+25}$

(function is neither odd nor even)

$\int_{-1}^{1} \frac{dx}{(x+4)^2+3^2}$

$\Rightarrow \frac{1}{3} \arctan\left(\frac{x+4}{3}\right)\Big|_{-1}^{1}$

$\Rightarrow \frac{1}{3} \arctan 5/3 - \frac{1}{3} \arctan 1$

$\Rightarrow = 1/3 \arctan 5/3 - \pi/12$

Consider:

$$\int \frac{u'}{\sqrt{a^2-u^2}}\,dx = \arcsin \frac{u}{a} + C$$

$$\int \frac{u'}{a^2+u^2}\,dx = \frac{1}{a} \arctan u/a + C$$

$$\int \frac{u'}{u\sqrt{u^2-a^2}}\,dx = 1/a \operatorname{arcsec} \frac{|u|}{a} + C$$

(because $x^2+8x+25 = x^2+8x+16+9$
$= (x+4)^2+3^2$)

6 Solve $\frac{dy}{dx} = \frac{1}{\sqrt{-x^2+2x}}$, $y(2) = \pi$

$\Rightarrow \quad y(x) = \int \frac{1}{\sqrt{-x^2+2x}}\,dx$

$$\left(\begin{aligned} -x^2+2x &= -(x^2-2x) \\ &= -(x^2-2x+1-1) \\ &= -((x-1)^2-1) \\ &= 1-(x-1)^2 \end{aligned} \right)$$

$\Rightarrow \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx = \arcsin(x-1) + C$

$a = 1$
$u(x) = x-1$

$\pi = y(2) = \arcsin(2-1) + C$
$= \arcsin 1 + C$
$= \pi/2 + C$
$\Rightarrow C = \pi - \pi/2 = \pi/2$
$= \quad y(x) = \arcsin(x-1) + \pi/2$

Lecture 5 - Integration and differentiation of inverse hyperbolic function (Section 5.9, continuation)

$\sinh^{-1} x$
$\cosh^{-1} x$
$\tanh^{-1} x$

Note: $\arcsin x = \sin^{-1} x \neq \cancel{1/\sin x} = (\sin x)^{-1}$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$t = e^x \Rightarrow \sinh x = \frac{t - 1/t}{2}$$

$$\Rightarrow 2y = t - 1/t$$
$$\Rightarrow 2y = t^2 - 1$$
$$\Rightarrow t^2 - 2yt - 1 = \emptyset$$
$$\Rightarrow t = \frac{2y \pm \sqrt{4y^2+4}}{2}$$
$$\Rightarrow t = y \pm \sqrt{y^2+1}$$
$$\Rightarrow e^x = y \pm \sqrt{y^2+1}$$
$$\Rightarrow e^x = y + \sqrt{y^2+1}$$
(because $e^x > 0$ and $y - \sqrt{y^2+1} < 0$)
$$\Rightarrow \sinh^{-1} x = \ln|x + \sqrt{x^2+1} \;???$$

Thm (inverse of hyperbolic functions)

Function	Domain
$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$	$(-\infty, \infty)$
$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$	$[1, \infty)$
$\tanh^{-1} x = 1/2 \ln\left(\frac{1+x}{1-x}\right)$	$(-1, 1)$
$\coth^{-1} x = 1/2 \ln\left(\frac{x+1}{x-1}\right)$	$(-\infty, -1) \cup (1, \infty)$
$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	$(0, 1]$
$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x \rvert}\right)$	$(-\infty, 0) \cup (0, \infty)$ $\mathbb{R} \setminus \{0\}$

Thm - Differentiation and integration involving inverse of hyperbolic functions.

Let $u(x)$ be a differentiable function.

Then

$$d/dx(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}} \qquad d/dx(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$$

$$d/dx(\tanh^{-1} x) = \frac{1}{1-x^2} \qquad d/dx(\coth^{-1} x) = \frac{1}{1-x^2}$$

$$d/dx(\operatorname{sech}^{-1} x) = \frac{1}{x\sqrt{1-x^2}} \qquad d/dx(\operatorname{csch}^{-1} x) = \frac{1}{\lvert x \rvert\sqrt{1+x^2}}$$

$$\int \frac{u'dx}{\sqrt{u^2 \pm a^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$$

$$\int \frac{u'dx}{a^2 - u^2} = \frac{1}{2a} \ln\left|\frac{a+u}{a-u}\right| + C$$

$$\int \frac{u'dx}{u\sqrt{a^2 \pm u^2}} = \frac{1}{a} \ln\left(\frac{a + \sqrt{a^2 \pm u^2}}{|u|}\right) + C$$

Examples

① $d/dx \left(\sinh^{-1}(x^2 + 2x)\right)$

$$\boxed{= \frac{1}{\sqrt{(x^2+2x)^2+1}} \cdot (2x+2)}$$

② $d/dx \left(\tanh^{-1}(\cos x)\right) = \dfrac{1}{1-\cos^2 x} \cdot (-\sin x)$

$$= \frac{1}{\sin^2 x} \cdot (-\sin x)$$

$$\boxed{= -\csc x}$$

③ $\displaystyle\int \frac{dx}{x\sqrt{9-25x^2}} \Rightarrow \int \frac{dx}{x\sqrt{3^2-(5x)^2}}$ $\quad a = 3$, $u(x) = 5x$

$$= \int \frac{5dx}{5x\sqrt{3^2-(5x)^2}} \Rightarrow \boxed{-\frac{1}{3} \ln\left(\frac{3 + \sqrt{9-25x^2}}{|5x|}\right) + C \quad C \in \mathbb{R}}$$

④ $\displaystyle\int \frac{dx}{3-2x^2} = \int \frac{dx}{(\sqrt{3})^2 - (\sqrt{2}x)^2}$ $\quad a = \sqrt{3}$, $u(x) = \sqrt{2}x$

$$= \frac{1}{\sqrt{2}} \int \frac{\sqrt{2}\,dx}{(\sqrt{3})^2 - (\sqrt{2}x)^2} = \frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{3}} \ln\left|\frac{\sqrt{3}+\sqrt{2}x}{\sqrt{3}-\sqrt{2}x}\right| + C$$

$$\boxed{= \frac{1}{2\sqrt{6}}}$$

⑤ $\displaystyle\int \frac{dx}{(x+3)(\sqrt{x^2+6x+12})} \Rightarrow \int \frac{dx}{(x+3)\sqrt{(\sqrt{3})^2 + (x+3)^2}}$

$$x^2 + 6x + 12 = x^2 + 6x + 9 - 3$$
$$= (x+3)^2 + (\sqrt{3})^2$$

$$\Rightarrow -\frac{1}{\sqrt{3}} \ln\left(\frac{\sqrt{3} + \sqrt{3+(x+3)^2}}{|x+3|}\right) + C$$

$C \in \mathbb{R}$

6 $\displaystyle\int \frac{1}{\sqrt{1+e^{2x}}}\,dx = \int \frac{1}{\sqrt{1^2+(e^x)^2}}\,dx \qquad a = 1, \quad u(x) = e^x$

$\displaystyle = \int \frac{e^x\,dx}{e^x\sqrt{1^2+(e^x)^2}} = -\ln\left(\frac{1+\sqrt{1+e^{2x}}}{e^x}\right) + C, \quad C \in \mathbb{R}$

Lecture 6 - Area of a region between two curves (section 7.1)

1. $f(x) = x^4 + x^2, \quad x \in [0, 1]$

Area $= \int_0^1 (x^4 + x^2)\,dx$

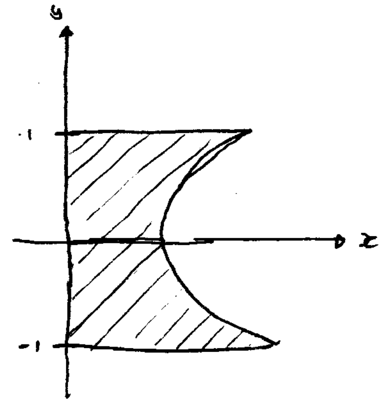
$$\Rightarrow \left(\frac{x^5}{5} + \frac{x^3}{3}\right)\Big|_0^1 \Rightarrow \frac{1}{5} + \frac{1}{3} = \frac{8}{15}$$

2. $f(y) = y^2 + 2, \quad y \in [-1, 1]$

or $x = y^2 + 2$

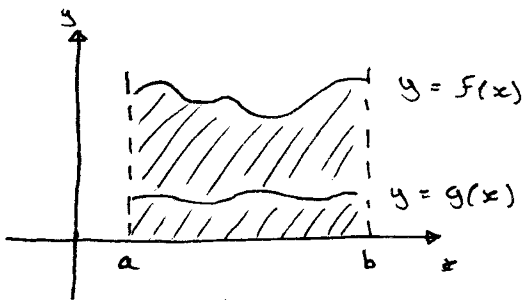
Area $= \int_{-1}^{1} (y^2+2)\,dy = \left(\frac{y^3}{3} + 2y\right)\Big|_{-1}^{1}$

$$\Rightarrow \tfrac{1}{3} + 2 - \left(\tfrac{1}{3} - 2\right) = \frac{14}{3}$$



$x = y^2 + 2$

Consider:



Known: ?
$f(x) \geq g(x)$

$$\text{Area} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

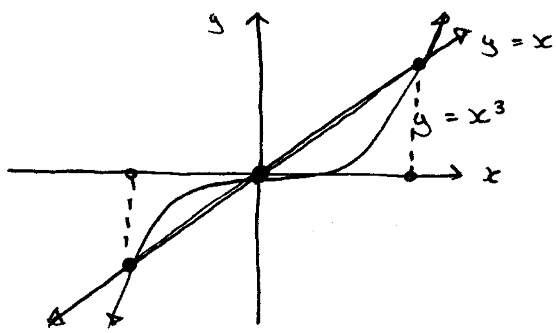
$$= \int_a^b (f(x) - g(x))\,dx \Rightarrow \int_a^b (\text{top curve} - \text{bottom curve})\,dx$$

Thm: (Area of region between two curves)
Let f and g be continuous functions on $[a, b]$ such that $f(x) \geq g(x)$ for all x on $[a, b]$. Then the area of the region bounded by $f(x)$, $g(x)$ and the lines $x = a$ and $x = b$ is given by: Area $= \int_a^b (f(x) - g(x))\,dx$

3. $f(x) = x^3 + 3 \qquad x \in [0, 1]$
$g(x) = -2x$

$$\text{Area} = \int_0^1 (f(x) - g(x))\,dx = \int_0^1 (x^3 + 3 - (-2x))\,dx$$
$$= \int_0^1 (x^3 + 2x + 3)\,dx$$
$$= \left(\frac{x^4}{4} + x^2 + 3x\right)\Big|_0^1 = 1/4 + 1 + 3$$
$$= \frac{17}{4}$$

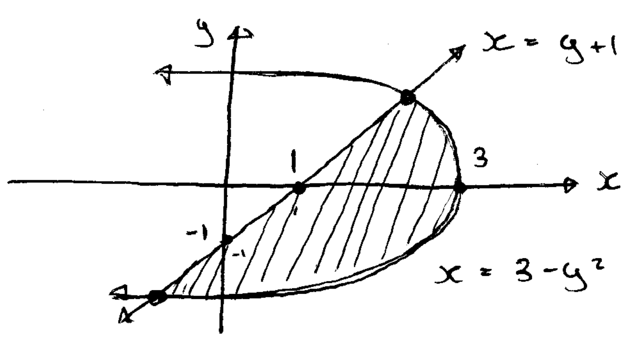
4. Find the area bounded by the curves $y = x^3$ and $y = x$



Sol. $x = x^3$
$\Leftrightarrow x^3 = x = 0$
$\Leftrightarrow x^3 - x = 0$
$\Leftrightarrow x(x-1)(x+1) = 0$
$\Leftrightarrow x = 0$
$x = 1$
$x = -1$

$$\text{Area} = \int_{-1}^{0} (x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx$$
$$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = -\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$$

5. Find the area of the region between the curves $x = 3 - y^2$ and $x = y + 1$



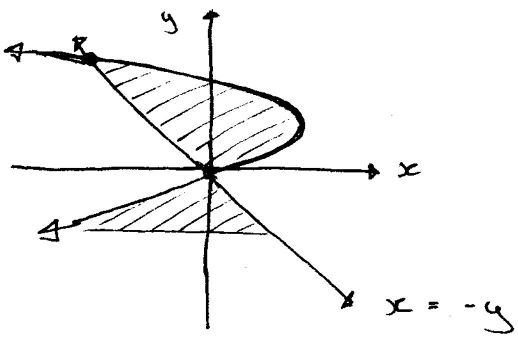
$$A = \int (3 - y^2 - (y+1))\,dy$$

Sol: $y + 1 = 3 - y^2$
$\Leftrightarrow y^2 + y - 2 = 0$
$\Leftrightarrow (y+2)(y-1) = 0$
$\Leftrightarrow y = -2, \quad y = 1$

$$A = \int_{-2}^{1} (3 - y^2 - (y+1))\,dy$$

$$= \left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_{-2}^{1} = 2 - \tfrac{1}{2} - \tfrac{1}{3} - \left(-4 - 2 + \tfrac{8}{3}\right) = 8 - \tfrac{1}{2} - 3 = \tfrac{9}{2}$$

(6) $f(y) = 2y - y^2$ $\quad y \in [-1, 3]$
$g(y) = -y$



$x = 2y - 2^2 = -(2y - 2^2)$
$\Rightarrow -(y^2 - 2y + 1 - 1)$
$\Rightarrow 1 - (y-1)^2$
vertex $(1, 1)$

$2y - y^2 = -y$
$\Rightarrow y^2 - 3y = 0$
$\Rightarrow y = 0, \quad y = 3$

Area