Lecture: • Power Series (Section 9.8)
• Representation of Functions by Power series (section 9.9)

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots \quad \text{Power series at } x=0$$

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots \quad \text{" " } x=c$$

Thm: There is $R \geq 0$ (R could be ∞) such that the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ conv. absolutely for $|x-c| < R$ ($c-R < x < c+R$) and diverges for $|x-c| > R$ ($x < c-R$ <u>OR</u> $x > c+R$)

R is the <u>radius of convergence</u>

Interval of convergence

The largest interval where the series converges.

<u>Example</u>

$$\sum_{n=1}^{\infty} \frac{1}{n}(x-1)^n$$

Let $a_n = \frac{1}{n}(x-1)^n$ Then

$a_{n+1} = \frac{1}{n+1}(x-1)^{n+1}$ and

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n+1}(x-1)^{n+1}}{\frac{1}{n}(x-1)^n}\right| = \lim_{n\to\infty} \frac{n}{n+1} |x-1| \frac{1/n}{1/n}$$

By the ratio test

The series conv. for $|x-c| < 1 \iff -1 < x < x-1 \iff 0 < x < 2$

" " div. " " $|x-1| > 1 \iff x > 2$ <u>OR</u> $x < 0$

<u>Radius of Conv.</u> $R = 1$

<u>Case $x = 2$</u> $\sum_{n=1}^{\infty} \frac{1}{n}(x-1)^n = \sum_{n=1}^{\infty} \frac{1}{n}$ (p-series $p=1$) Harmonic Series

↳ Diverges

<u>Case $x = 0$</u> $\sum_{n=1}^{\infty} \frac{1}{n}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ Alternating Series Test

↳ conv.

Let $a_n = 1/n$

$$\left(\begin{array}{l} ① \ a_n \geq 0 \\ ② \ \lim_{n\to\infty} a_n = 0 \\ ③ \ \{a_n\} \text{ decreases} \end{array} \right)$$

Thm (Properties of Power Series)

If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has radius of convergence

$R > 0$, then f is differentiable for $|x-c| < R$

and ① $f'(x) = \sum_{n=0}^{\infty} a_n (x-c)^{n-1}$

② $\int f'(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}\left((x-c)^{n+1} + 1\right)$

(where $1 \in \mathbb{R}$)

Moreover, the radius of conv. of both series is also R.

Example:

Find $f(x)$ such that:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Solution $\sum^{\infty} \frac{x^n}{n!}$ conv. for all x (i.e. $R = \infty$)

We have $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

and $f'(x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = f(x)$

So, $f'(x) = f(x)$ for all x

and for $f(0) = 1$

$\Downarrow$

Hence,

$$(\ln f(x))' = \frac{f'(x)}{f(x)} = 1$$

$$\Rightarrow \ln f(x) = \int dx = x + c$$

$$\Rightarrow f(x) = e^{x+c}$$

Since $f(0) = 1$, we get $1 = f(0) = e^{0+c} = e^c$

So, $e^c = 1$

Therefore $f(x) = e^{x+c} = e^x \cdot e^c = e^x \cdot 1 = e^x$

So, $f(x) = e^x$ and

$$e^x = f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

<u>Example</u> Solve $y' = y$, $y(0) = 1$ using power series

<u>Solution</u> Suppose that $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$

Then $y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$

Since $y' = y$ we get:

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots = a_0 + a_1 x + a_2 x^2 + \dots$$

So,

$$\left.\begin{aligned} a_1 &= a_0 \\ 2a_2 &= a_1 \\ 3a_3 &= a_2 \\ 4a_4 &= a_3 \end{aligned}\right\} \rightarrow \begin{aligned} a_1 &= a_0 \\ a_2 &= \tfrac{1}{2} a_1 = \tfrac{1}{2!} a_0 \\ a_3 &= \tfrac{1}{3} a_2 = \tfrac{1}{3} \cdot \tfrac{1}{2!} a_0 = \tfrac{1}{3!} a_0 \\ a_4 &= \tfrac{1}{4} a_3 = \tfrac{1}{4} \cdot \tfrac{1}{3!} a_0 = \tfrac{1}{4!} a_0 \\ &\vdots \\ a_n &= \tfrac{1}{n!} a_0 \end{aligned}$$

So, $y(x) = a_0 + a_1 x \cdots = a_0 + a_0 x + a_0 \frac{x^2}{2!}$

$= a_0 \left(1 + x + x^2/2! + x^3/3! + \dots\right)$

$= a_0 e^x$

Since $y(0) = 1$, we get

$1 = y(0) = a_0 e^0 = a_0 \Rightarrow a_0 = 1$

Therefore $y = e^x$

Geometric Power Series

$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$

Radius of conv. : $R = 1$

Interval " " : $(-1, 1)$

Example :

Find the power series of

$f(x) = 1/1-x$ at $x = 2$

Sol. We want to find $a_0, a_1, a_2, \ldots$

Such that : $\frac{1}{1-x} = f(x) = \sum_{n=0}^{\infty} a_n (x-2)^n$

We have :

$$\frac{1}{1-x} = \frac{1}{1-(x-2)-2} = \frac{1}{-1-(x-2)} = \frac{-1}{1+(x-2)}$$

$$= -\sum_{n=0}^{\infty} (-(x-2))^n \Rightarrow -\sum_{n=0}^{\infty} (-1)^n (x-2)^n = \sum_{n=0}^{\infty} (-1)^{n+1} (x-2)^n$$

Thm (Properties of Power)

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and

$g(x) = \sum_{n=0}^{\infty} b_n x^n$

Then

1. $f(Hx) = \sum_{n=0}^{\infty} a_n H^n x^n$

2. $f(x^N) = \sum_{n=1}^{\infty} a_n x^{nN}$

3. $f(x) + g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

Example

Find the power series of $f(x) = \arctan x$ at $x = 0$

We have:

$$f'(x) = \frac{1}{1+x^2} \quad \text{and} \quad g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{For } |x| < 1$$

Then

$$f'(x) = g(-x^2) = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} - |x| < 1$$

LAB - SERIES

Geometric Series

1. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$

$r = \frac{1}{\sqrt{2}}$ (converges)

$a \neq 0$

$$\sum_{n=1}^{\infty} ar^n = a + ar^0 + ar^2 + \ldots = \begin{cases} \frac{a}{1-r} & |r|<1 \\ \text{div.} & |r| \geq 1 \end{cases}$$

$|r| < 1$

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n - \left(\frac{1}{\sqrt{2}}\right)^0 = \frac{1}{1-\frac{1}{\sqrt{2}}} - 1$$

$a = 1$

$$= \frac{1}{\sqrt{2}-1}$$

2. $\sum_{n=1}^{\infty} 2^{2n} \cdot 3^{1-n}$

$$\Rightarrow \sum_{n=1}^{\infty} 4^n \cdot 3 \cdot 3^{-n}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{4^n}{3^n} \cdot 3 \Rightarrow \sum_{n=1}^{\infty} 3\left(4/3\right)^n$$

$r = 4/3 \quad |r| > 1$ (diverges)

TELESCOPING:

1. $\sum_{n=1}^{\infty} \left(\cos\left(\frac{1}{n+1}\right) - \cos\left(\frac{1}{n}\right)\right)$

$$= \lim_{H \to \infty} S_H$$

$$S_H = \sum_{H \to 0}^{H} \left(\cos\left(\frac{1}{n+1}\right) - \cos\left(\frac{1}{n}\right)\right)$$

and he's doing something different...

$$\sum_{n=0}^{\infty} (a_{n+1} - a_n) = \lim_{H \to \infty} S_H$$

$$S_H = \sum_{H=0}^{H} (a_{n+1} - a_n) = a_1 - a_0 + a_2 - a_1 + \ldots + a_n - a_{n-1} + a_{n+1} - a_n$$

$$= a_{n+1} - a_0$$

$$\sum_{n=0}^{\infty} (a_{n+1} - a_n) = \lim_{n \to \infty} (a_{n+1} - a_0)$$

$$= \lim_{H \to \infty} S_H$$

His handwriting is just awful.

Let $a_n = \cos\left(\frac{1}{n}\right)$

Then $a_{n+1} = \cos\left(\frac{1}{n+1}\right)$

$$\sum_{n=1}^{\infty}\left(\cos\left(\tfrac{1}{n+1}\right) - \cos\left(\tfrac{1}{n}\right)\right) = \lim_{n\to\infty}\left(\cos\left(\tfrac{1}{n+1}\right) - \cos\left(\tfrac{1}{1}\right)\right)$$

$$= \cos 0 - \cos(1)$$

$$= 1 - \cos 1$$

Convergent

$$S_K = \sum_{n=1}^{\infty}\left(\cos\left(\tfrac{1}{n+1}\right) - \cos\left(\tfrac{1}{1}\right)\right)$$

$$= \cancel{\cos \tfrac{1}{2}} - \cos \tfrac{1}{1} + \cancel{\cos \tfrac{1}{3}} - \cancel{\cos \tfrac{1}{2}} \dots$$
$$+ \cos\left(\tfrac{1}{A+1}\right) - \cancel{\cos\left(\tfrac{1}{A}\right)}$$

$$= -\cos 1 + \cos\left(\tfrac{1}{A+1}\right)$$

(2) $$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty}\left(\ln n - \ln(n+1)\right)$$
$$= -\sum_{n=1}^{\infty}\left(\ln(n+1) - \ln n\right)$$

$$= -\lim_{n\to\infty}\left(\ln(n+1) - \ln 1\right) = -\infty \quad \text{div.}$$

(3) $$\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2} = \sum_{n=0}^{\infty} \frac{1}{(n+2)(n+1)}$$

$$\frac{1}{(n+2)(n+1)} = \frac{A}{n+2} + \frac{B}{n+1}$$

$A = -1$

$B = 1$

$$\Rightarrow \frac{-1}{n+2} + \frac{1}{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = -\sum_{n=0}^{\infty}\left(\frac{1}{n+1+1} - \frac{1}{n+1}\right)$$

$$\Rightarrow -\lim_{n\to\infty}\left(\frac{1}{n+1+1} - \frac{1}{0+1}\right) = 1 \quad \text{conv.}$$

<u>n^{th} term div. test</u> $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n$ div.

1. $\sum_{n=1}^{\infty} n^2 \left(1 - \cos(1/n)\right) \Rightarrow \infty \cdot 0$

$$\lim_{n\to\infty} n^2 \left(1 - \cos(1/n)\right)$$

$$\lim_{n\to\infty} \frac{1 - \cos(1/n)}{1/n^2} \Rightarrow \lim_{x\to\infty} \frac{1 - \cos(1/x)}{1/x^2} = \left(\frac{0}{0}\right)$$

$$\lim_{n\to\infty} \frac{1 - \cos(1/x)}{1/x^2} \Rightarrow \lim_{x\to\infty} \frac{\sin(1/x)\cdot(-1/x^2)}{-2/x^3}$$

$$\Rightarrow \lim_{x\to\infty} \frac{\sin(1/x)}{2/x}$$

$$= \lim_{x\to\infty} \frac{\cos(1/x)\cdot(-1/x^2)}{(-2/x^2)} = 1/2$$

So, $\sum_{n=1}^{\infty} n^2 \left(1 - \cos 1/n\right)$ div.

Lecture · Representation of Functions by power series (9.9)
· Taylor and Maclaurin Series

$1 + y + y^2 + \dots = \frac{1}{1-y}$ for $|y| < 1$
$(-1 < y < 1)$

Example:

1. $f(x) = \frac{1}{x^2 - 1}$ At $x = 0$

$$= \frac{-1}{1 - x^2} = -\sum_{n=0}^{\infty} (x^2)^n = -\sum_{n=0}^{\infty} x^{2n} \quad (|x| < 1)$$

2. $f(x) = \frac{1}{(1+x)^2} = \left(-\frac{1}{1+x}\right)'$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\Rightarrow -\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^{n+1} x^n$$

Taking derivative, we get

$$f(x) = \frac{1}{(1+x)^2} = -\left(\frac{1}{1+x}\right)' = \sum_{n=0}^{\infty} (-1)^{n+1} (x^n)'$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1} \quad (-1 < x < 1)$$

3. $f(x) = \ln x$ at $x = 1$ $\left(f(x) = \sum_{n=1}^{\infty} a_n (x-1)^n\right)$

$\ln x + C = \int \frac{1}{x}\, dx$

$$\frac{1}{x} = \frac{1}{1 + x - 1} = \frac{1}{1 - (-(x-1))} = \sum_{n=0}^{\infty} (-(x-1))^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$y = -(x-1)$ for $|x-1| < 1$

Integrating, we get

Integrating we get

$$\ln x + C = \int 1/x \, dx = \sum_{n=0}^{\infty} (-1)^n \int (x-1)^n dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$

$$\Rightarrow \ln x + C = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} \Rightarrow C = \ln 1 + C = \sum_{n=0}^{\infty} (-1)^n \frac{(1-1)^{n+1}}{n+1} = 0$$

$$\Rightarrow \ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{(n+1)}$$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

For $x = 0$ $\quad f(0) = a_0$

$a_0 = f(0)$ $\quad f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$

For $x = 0$ $\quad f'(0) = a_1$

$a_1 = f'(0)$ $\quad f^2(x) = 2a_2 + 3! a_3 x + \ldots$

For $x = 0$ $\quad f^2(0) = 2a_2$

$a_2 = f^2(0)/2$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$$\Rightarrow f(x) = f(0) + \frac{f'(0)}{1!} + \frac{f^2(0)}{2!} + \ldots$$

thm. (The Form of a convergent power series)
If f is represented by a power series $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ for all x in an open interval that contains c, then:

$$a_n = \frac{f^{(n)}(C)}{n!}$$

So, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} x^n$

<u>Def'n</u> (Taylor and Maclaurin Series)

If f has derivatives of all orders at c, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)(x-c)^n}{n!}$$

is called the Taylor Series for $f(x)$ at c. if $c = 0$ then the series is the Maclaurin Series of f.

<u>Remark</u>

The n^{th} Taylor polynomial of f is the n^{th} partial sum of the taylor series. (Same for Maclaurin Polynomials)

<u>Example</u> (compute the 5^{th} Maclaurin Polynomial)

$f(x) = \frac{1}{(1+x)^2}$

Solution :

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1} = 1 - 2x + 3x^2 - 4x^3 + 5x^4$$

$$Q_5(x) = 1 - 2x + 3x^2 - 4x^3 + 5x^4$$

<u>Thm</u> (Convergence of Taylor Series)

If $\lim_{n\to\infty} R_n(x) = 0$, where $R_n(x) = \frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!}$

Then the taylor series of f at c converges to f. That is :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Proof :

$f(x) = P_n(x) + R_n(x)$

$$\sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

(n^{th} taylor pol.)

"

(n^{th} partial sum of the Taylor series)

If $R_n(x) \longrightarrow 0$, then $P_n(x) \rightarrow f(x)$

Examples :

① $e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \forall x$

② $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

Thm (Binomial Formula)

If k is not a positive integer and $k \neq 0$

Then

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \ldots + \frac{k(k-1)\,(k-n+1)}{n!}x^n + \ldots$$

For $-1 < x < 1$

Examples :

① $(1+x)^{1/2} = 1 + 1/2\, x + \frac{1/2\,(1/2 - 1)}{2!}x^2 + \frac{1/2\,(1/2-1)(1/2-1)}{3!}x^3 + \ldots$

$R = 1/2 \qquad -1 < x < 1$

Multiplication of Series

$f(x) = e^x \sin x$ $\qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$\sin x = x - x^3/3! + x^5/5! + \dots$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$+ x^2 - \frac{x^4}{5!} + \frac{x^6}{5!} - \dots$$

etc

$$= x + x^2 + \left(1/2! - 1/3!\right)x^3 + \dots$$

Division

$f(x) = \frac{\sin x}{\cos x} = \tan x$ $\qquad \sin x = x - \frac{x^3}{3!} + \dots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \overline{\Big) x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}$$

$$- \left(x - \frac{x^3}{2!} + \frac{x^5}{4!} - \dots\right)$$

$$\overline{(1/2! - 1/3!)x^3 (1/5! - 1/4!)x^5 + \dots}$$

END OF LECTURE NOTES.

For exam:

5.5, 7.1 - 7.4, Chapter 8 (Not 8.6), Chapter 9

Series:

Computing the sum of series

- Geometric series: $\displaystyle\sum_{n=0}^{\infty} ar^n \begin{cases} \frac{a}{1-r} & \text{if } |r|<1 \\ \text{div} & \text{if } |r| \geq 1 \end{cases}$
- Telescoping series: $\displaystyle\sum_{n=0}^{\infty} (a_{n+1} - a_n) = \lim_{n\to\infty} (a_{n+1} - a_0)$

$$\sum_{n=0}^{\infty} (a_n - a_{n+1}) = \lim_{n\to\infty} (a_0 - a_{n+1})$$

(Geometric Series)

1 Examples:

$$\sum_{n=1}^{\infty} \frac{2^{1-n} + 3^{2n}}{10^{n+1}} \Rightarrow \sum_{n=0}^{\infty} \frac{2^{1-n} + 3^{2n}}{10^{n+1}} - \frac{2^{1-0} + 3^{2(0)}}{10^{(0)+1}}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{\frac{2}{2^n} + 9^n}{10 \cdot 10^n} - \frac{3}{10} = \sum_{n=0}^{\infty} \left(\frac{2}{10} \cdot \frac{1}{2^n 10^n} + \frac{1}{10} \cdot \frac{9^n}{10^n} \right) - \frac{3}{10}$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(\frac{2}{10}\left(\frac{1}{20}\right)^n + \frac{1}{10}\left(\frac{9}{10}\right)^n \right) - \frac{3}{10} = \frac{\frac{1}{5}}{1 - \frac{1}{10}} + \frac{\frac{1}{10}}{1 - \frac{9}{10}} - \frac{3}{10}$$

$a = 1/5$ $\quad$ $a = 1/10$

$r = 1/20 < 1$ $\quad$ $r = 1/10 < 1$

(Telescoping)

2 Example:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 5n + 6} \Rightarrow \frac{1}{(n+2)(n+3)} = \frac{A}{(n+2)} + \frac{B}{(n+3)}$$

$$\Rightarrow 1 = A(n+3) + B(n+2) = (A+B)n + 3A + 2B$$

$$\Rightarrow \begin{array}{l} A + B = 0 \\ 3A + 2B = 0 \end{array} \Rightarrow \begin{array}{l} A = 1 \\ B = -1 \end{array}$$

$$\Rightarrow \frac{1}{n^2 + 5n + 6} = \frac{1}{n+2} - \frac{1}{n+3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+5n+6} = \sum_{n=1}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n+3} \right)$$

$$= \sum_{n=1}^{\infty} (a_n - a_{n+1}) = \lim_{n\to\infty} (a_1 - a_{n+1})$$

Let $a_n = \frac{1}{n+2}$. Then $a_{n+1} = \frac{1}{n+2+1} = \frac{1}{n+3}$

$$\lim_{n\to\infty} \left(\frac{1}{3} - \frac{1}{n+3}\right) = 1/3$$

$a_1 = 1/_{1+2} = 1/3$ conv.

Convergence Tests

Examples :

① $\sum_{n=0}^{\infty} ne^{-n} \Rightarrow \lim_{n\to\infty} ne^{-n} = \lim_{n\to\infty} \frac{n}{e^n} = 0$ $(\frac{\infty}{\infty})$

THEREFORE

$$= \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

Then the test is inconclusive.

② $\sum_{n=0}^{\infty} ne^{-1/n} \Rightarrow \lim_{n\to\infty} ne^{-1/n} = \infty$ $(\infty \cdot 1)$

$\neq 0$

Then it is divergent.

n^{th} term divergence test $\left(\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum a_n \text{ div}\right)$

P - Series $\left(\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{array}{l} \text{conv. if } p > 1 \\ \text{div if } p \le 1 \end{array} \right)$

Example:

① $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n\sqrt[3]{n}} \Rightarrow \sum_{n=1}^{\infty} \frac{n^{1/2}}{n \cdot n^{1/3}} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{1+1/3-1/2}}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{5/6}} \Rightarrow P = 5/6 < 1$

div.

2 Integral Test
$$\left(\begin{array}{l} \sum a_n, \quad a_n = f(n) \\ f(x) \geq 0 \quad \text{on } [1, \infty) \\ f(x) \text{ decreases } \_\_\_\_ \end{array} \right.$$

Then $\sum a_n$ Conv. if and only if $\int_1^\infty f(x)\,dx$ conv.

Examples:

1 $\sum_{n=1}^{\infty} ne^{-n}$

Let $f(x) = \cancel{x}e^{-x}$ Then

1 $f(x) = xe^{-x} = f(n) = ne^{-n}$

2 $f(x) \geq 0$ for $x \geq 0$

3 $f'(x) = e^{-x} + x(-e^{-x}) = e^{-x}(1-x)$

$f'(x) \leq 0 \iff 1-x \leq 0 \iff x \geq 1 \iff x \in [1,\infty)$

$\Rightarrow f$ decreases on $[1, \infty)$

$$\int_1^\infty f(x)dx = \int_1^\infty xe^{x}dx = \lim_{b\to\infty} \int_1^b xe^{-x}dx$$

$$= \lim_{b\to\infty} \left(-xe^{-x} - e^{-x}\right)\Big|_1^b = \lim_{b\to\infty}\left(-be^{-b} - e^{-b} - e^{-1} - e^{-1}\right)$$

$$\Rightarrow -0 - 0 + 2e^{-1} = 2e^{-1} \quad \underline{\text{Conv.}}$$

$$\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x}$$

$u = x \Rightarrow u' = 1$

$v' = e^{-x} \Rightarrow v = -e^{-x}$

$$\lim_{b\to\infty} e^{-b} \Rightarrow \lim_{b\to\infty} \frac{1}{e^b} = 0, \quad \lim_{b\to\infty} e^{-b} = \lim_{b\to\infty} \frac{b}{e^b}$$

$$= \lim_{b\to\infty} \frac{1}{e^b}$$

$= 0$???

wtf

~~Xxxxxx~~ Direct Comparison Test

$$\left(\begin{array}{l} \sum a_n,\ \sum b_n \\ 0 \le a_n \le b_n \\ \quad \sum b_n \text{ conv.} \Rightarrow \sum a_n \text{ conv.} \\ \quad \sum a_n \text{ div} \Rightarrow \sum b_n \text{ div} \end{array} \right)$$

Examples

① $\displaystyle\sum_{n=1}^{\infty} \frac{2-\cos n}{n^2}$

$$\left[\begin{array}{l} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \\ p = 2 > 1 \\ \therefore \text{conv.} \end{array} \right]$$

$$a_n = \frac{2-\cos n}{n^2}$$

$$-1 \le -\cos n \le 1$$

$$1 \le 2-\cos n \le 3$$

$$\Rightarrow \quad 0 \le \frac{1}{n^2} \le \underbrace{\frac{2-\cos n}{n^2}}_{a_n} \le \underbrace{\frac{3}{n^2}}_{b_n}$$

$$\sum b_n = \sum \frac{3}{n^2} = 3 \sum \frac{1}{n^2}$$

$\therefore$ conv.

Limit Comparison Test

$$\left(\begin{array}{l} a_n,\ b_n \to 0 \\ \displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0, \text{ Finite} \end{array} \right.$$

Then $\underset{\text{conv.}}{\sum a_n}$ if and only if $\underset{\text{conv.}}{\sum b_n}$

Example:

① $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n^2} \Rightarrow a_n = \frac{\sqrt{n^2+1}}{n^2} \qquad b_n = \frac{\sqrt{n^2}}{n^2} = \frac{n}{n^2} = \frac{1}{n}$

&

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n^2+1}}{n^2}}{1/n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n} = \lim_{n\to\infty} \sqrt{\frac{n^2+1}{n^2}}$$

$$= \lim_{n\to\infty} \sqrt{1+1/n^2} = 1 > 0$$

$$\sum b_n = \sum \frac{1}{n}$$

$$p = 1 \le 1$$

divergent.